

FREE ALGEBRAIC STRUCTURES ON THE PERMUTOHEDRA

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ABSTRACT. Tridendriform algebras are a type of associative algebras, introduced independently by F. Chapoton and by J.-L. Loday and the third author, in order to describe operads related to the Stasheff polytopes. The vector space \mathcal{ST} spanned by the faces of permutohedra has a natural structure of tridendriform bialgebra, we prove that it is free as a tridendriform algebra and exhibit a basis. Our result implies that the subspace of primitive elements of the coalgebra \mathcal{ST} , equipped with the coboundary map of permutohedra, is a free cacti algebra.

INTRODUCTION

The graded vector space spanned by the set of planar rooted trees has a rich algebraic structure, coming from the different set-theoretical operations which can be performed on trees. Furthermore, when we consider the set of planar rooted trees \mathcal{T}_n with a fixed number n of leaves, it has a natural structure of partially ordered set whose geometric realization is a polytope of dimension $n - 1$, the Stasheff polytope.

Trendriform algebras were defined independently by F. Chapoton in [3] and J.-L. Loday and the third author in [8], in order to generalize the notion of dendriform algebra introduced by J.-L. Loday in [7], and to get a non-symmetric operad structure described in terms of the faces of the Stasheff polytopes. The definitions are similar, even if they do not coincide, Chapoton's one is the graded version of Loday-Ronco's tridendriform version. In previous work of the first and third authors [2], the two notions were described in the same framework, by adding a parameter q : Chapoton's operad coincides with the notion of 0-tridendriform, while one recovers the original definition of tridendriform algebra of Loday and Ronco for $q = 1$.

Trendriform algebras are a particular type of non-unital associative algebras, where the associative product is the sum of certain binary operations. Many examples of associative algebras arising from combinatorial Hopf algebras, as the bialgebra of surjective maps (see [12] and [15]), the bialgebra

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of parking functions (see [12]) and the bialgebra of multipermutations (see [6]), come from 1-tridendriform structures. However, the 0-tridendriform version has the advantage of giving the right definition for working in the graded differential case, as shown by F. Chapoton in [3], who gave a version of differential graded tridendriform operad which is described by the Stasheff polytope.

In [2], the first and the third authors also defined q -Gerstenhaber-Voronov algebras as associative algebras equipped with an additional brace structure, satisfying certain relations. When $q = 0$, these algebras may be considered a non-differential version of the operad defined in [4]. They proved that the category of conilpotent q -tridendriform bialgebras is equivalent to the category of q -Gerstenhaber-Voronov algebras, via the functor which associates to any coalgebra the subspace of its primitive elements.

As a direct consequence of this result, we get that proving the freeness of a tridendriform algebra A is equivalent to proving that the subspace of its primitive elements $\text{Prim}(A)$ is free as a Gerstenhaber-Voronov algebra, when A is a tridendriform bialgebra.

The work presented here deals with the dendriform and tridendriform bialgebra structures defined on the space spanned by the faces of permutohedra. The permutohedron of dimension $n - 1$ is a regular polytope which is the geometric realization of the Coxeter poset of the symmetric group S_n . Its faces of dimension r are described by the surjective maps from $\{1, \dots, n\}$ to $\{1, \dots, n - r\}$, for $0 \leq r \leq n - 1$. The vector space spanned by the faces of all permutohedra has natural structures of dendriform and q -tridendriform bialgebras, denoted \mathcal{ST}_D and \mathcal{ST}_{qT} respectively.

The goal of our work is twofold: we construct a basis of \mathcal{ST}_D as a free dendriform algebra, and a basis of \mathcal{ST}_{qT} as a free q -tridendriform algebra. We show that the subspace $\text{Prim}(\mathcal{ST})$ of primitive elements of the coalgebra \mathcal{ST} is a free brace algebra for the brace structure induced by \mathcal{ST}_D . Then we proceed and construct a basis \mathcal{B} of $\text{Prim}(\mathcal{ST})$ as a free q -Gerstenhaber-Voronov algebra. The last result implies that:

- (1) \mathcal{B} is a basis of the free q -tridendriform algebra \mathcal{ST}_{qT} , for all q ;
- (2) looking at the graded differential case, that $\text{Prim}(\mathcal{ST})$ is the free cacti algebra spanned by \mathcal{B} .

In a recent work V. Vong, see [16] describes combinatorial methods to study the freeness of some algebras over regular operads. Our method is essentially different, our proof relies on the following outline:

- (1) the vector space $\text{Prim}(\mathcal{ST})$ is isomorphic to the vector space $\mathbb{K}[\mathbf{Irr}]$, spanned by the irreducible elements of \mathbf{ST} for the concatenation product \times
- (2) there exist surprisingly simple ways to define a free brace structure, respectively a free q -Gerstenhaber-Voronov algebra structure, on the space $\mathbb{K}[\mathbf{Irr}]$,

- (3) the brace algebra $\text{Prim}(\mathcal{ST}_D)$ is isomorphic to $\mathbb{K}[\mathbf{Irr}]$ with its brace algebra structure, while $\mathbb{K}[\mathbf{Irr}]$ with its q -Gerstenhaber-Voronov algebra structure is isomorphic to $\text{Prim}(\mathcal{ST}_{qT})$.

We hope that this type of process will provide a standard method to define new brace, q -Gerstenhaber-Voronov and cacti structures related to combinatorial Hopf algebras.

The paper is composed as follows: the first two sections recall basic results on coalgebras, as well as the definitions of (tri)dendriform algebras and Gerstenhaber-Voronov algebras, and the main results about tridendriform bialgebras which we use hereinafter.

In section 3 we give the basic definitions and constructions on surjective maps needed in later sections.

Section 4 describes the coalgebra structure of \mathcal{ST} , as well as a projection from \mathcal{ST} onto the space of its primitive elements.

In section 5 we describe the dendriform structure of \mathcal{ST}_D , we define a free brace algebra structure on the space spanned by the set of irreducible surjections and we prove that it is isomorphic to the brace algebra $\text{Prim}(\mathcal{ST}_D)$. Finally, in sections 6 through 8, we prove a similar result for the q -tridendriform algebra \mathcal{ST}_{qT} .

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Notation. All the vector spaces considered in the present work are over \mathbb{K} , where \mathbb{K} is a field. For any set X , we denote by $\mathbb{K}[X]$ the vector space spanned by X . For any \mathbb{K} -vector space V , we denote by $V^+ := \mathbb{K} \oplus V$ the augmented vector space.

1. COALGEBRAS

We recall the definition of coalgebra, and introduce the notation and basic results that we need in the rest of the work.

1.1. Definition. A *coalgebra* over \mathbb{K} is a vector space C equipped with a linear map $\Delta : C \longrightarrow C \otimes C$ which satisfies the coassociativity condition:

$$(\Delta \otimes Id_C) \circ \Delta = (Id_C \otimes \Delta) \circ \Delta.$$

An augmentation of a coalgebra (C, Δ) is a linear map $\epsilon : C \longrightarrow \mathbb{K}$ such that $\cdot \circ (\epsilon \otimes Id_C) = Id_C = \cdot \circ (Id_C \otimes \epsilon)$, where \cdot denotes the action of \mathbb{K} on C .

A unit of a coalgebra (C, Δ, ϵ) is a coalgebra map $\iota : \mathbb{K} \longrightarrow C$ such that $\epsilon \circ \iota = Id_{\mathbb{K}}$, where the coalgebra structure of \mathbb{K} is given by $\Delta_{\mathbb{K}}(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$.

1.2. Definition. Let (C, Δ) be a unital augmented coalgebra. An element $c \in C$ is *primitive* if $\Delta(c) = c \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes c$. The *reduced coproduct* on C is defined as

$$\overline{\Delta}(c) := \Delta(c) - 1_{\mathbb{K}} \otimes c - c \otimes 1_{\mathbb{K}}.$$

We denote by $\text{Prim}(C)$ the subspace of primitive elements of C .

It is immediate to verify that the coassociativity of Δ implies that $\overline{\Delta}$ is coassociative, too. We define $\overline{\Delta}^i : C \longrightarrow C^{\otimes i}$ recursively by:

- (1) $\overline{\Delta}^1 := Id_C$ is the identity of C ,
- (2) $\overline{\Delta}^i := (Id_{C^{\otimes i-1}} \otimes \overline{\Delta}) \circ \overline{\Delta}^{i-1}$, for $i \geq 2$.

1.3. Definition. A coassociative counital and unital coalgebra (C, Δ) is called *conilpotent* if for all $c \in \overline{C}$, there exists $n \in \mathbb{N}$ such that $\overline{\Delta}^m(c) = 0$, for all $m \geq n$.

1.4. Example. Let V be a \mathbb{K} -vector space. The vector space $T(V) := \bigoplus_{n \geq 1} V^{\otimes n}$, where $V^{\otimes n}$ denotes the tensor product $V \otimes V \otimes \dots \otimes V$ of V n -times, equipped with the deconcatenation coproduct:

$$\Delta^c(v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes \dots \otimes v_n),$$

is a coalgebra. We denote it $T^c(V)$, and call it the *cotensor coalgebra* over V .

Note that $T^c(V)^+$ is a unital augmented conilpotent coalgebra.

2. DENDRIFORM AND TRIDENDRIFORM BIALGEBRAS

We recall basic results on dendriform and tridendriform bialgebras (see [7], [3] and [8]), and Gerstenhaber-Voronov algebras (see [4]). We also describe the main results of [2].

2.1. Definition. Let A be a vector space over \mathbb{K} .

- (1) A *dendriform algebra* (see [7]) structure on A is a pair of binary products $\prec : A \otimes A \rightarrow A$ and $\succ : A \otimes A \rightarrow A$, satisfying that:
 - (a) $(a \prec b) \prec c = a \prec (b \prec c + b \succ c)$,
 - (b) $(a \succ b) \prec c = a \succ (b \prec c)$,
 - (c) $(a \prec b + a \succ b) \succ c = a \succ (b \succ c)$.
- (2) For any $q \in \mathbb{K}$, a *q-tridendriform algebra* structure on A is given by three binary operations $\prec : A \otimes A \rightarrow A$, $\cdot : A \otimes A \rightarrow A$ and $\succ : A \otimes A \rightarrow A$, which satisfy the following relations:

- (a) $(a \prec b) \prec c = a \prec (b \prec c + b \succ c + q b \cdot c)$,
- (b) $(a \succ b) \prec c = a \succ (b \prec c)$,
- (c) $(a \prec b + a \succ b + q a \cdot b) \succ c = a \succ (b \succ c)$,
- (d) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- (e) $(a \succ b) \cdot c = a \succ (b \cdot c)$,
- (f) $(a \prec b) \cdot c = a \cdot (b \succ c)$,
- (g) $(a \cdot b) \prec c = a \cdot (b \prec c)$.

If (A, \prec, \cdot, \succ) is a q -tridendriform algebra, then the space A equipped with the binary operations \prec and $\succ := q \cdot + \succ$ is a dendriform algebra. On the other hand, for any dendriform algebra, the operation $* = \succ + \prec$ is associative. So, dendriform and q -tridendriform algebras are particular cases of non-unital associative algebras.

2.2. Definition. A *dendriform bialgebra* over \mathbb{K} is a dendriform algebra (H, \succ, \prec) equipped with a coassociative coproduct $\Delta : H^+ \rightarrow H^+ \otimes H^+$ and a counit $\epsilon : H^+ \rightarrow \mathbb{K}$ satisfying the following conditions:

- (1) $(\epsilon \otimes Id) \circ \Delta(x) = 1_{\mathbb{K}} \otimes x$ and $(Id \otimes \epsilon) \circ \Delta(x) = x \otimes 1_{\mathbb{K}}$,
- (2) $\Delta(x \succ y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \succ y_{(2)})$,
- (3) $\Delta(x \prec y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \prec y_{(2)})$,

for all $x, y \in H$ where $* = \succ + \prec$, $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, and by convention:

- $(x * y) \otimes (1_{\mathbb{K}} \succ 1_{\mathbb{K}}) := (x \succ y) \otimes 1_{\mathbb{K}}$,
- $(x * y) \otimes (1_{\mathbb{K}} \prec 1_{\mathbb{K}}) := (x \prec y) \otimes 1_{\mathbb{K}}$, for $x, y \in H$.

A *q -tridendriform bialgebra* is a q -tridendriform algebra H with a coproduct Δ such that:

- (1) $(H, \succ, \cdot, \prec, \Delta)$ is a dendriform bialgebra,
- (2) $\Delta(x \cdot y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \cdot y_{(2)})$,

where $(x * y) \otimes (1_{\mathbb{K}} \cdot 1_{\mathbb{K}}) := (x \cdot y) \otimes 1_{\mathbb{K}}$.

We observe that if $(H, \succ, \cdot, \prec, \Delta)$ is a q -tridendriform bialgebra, then $(H, \succ, \prec, \Delta)$ is a dendriform bialgebra and $(H^+, *, \Delta^+)$ is a bialgebra in the usual sense.

For any bialgebra H , the subspace $\text{Prim}(H)$ has a natural structure of Lie algebra, but in the case of dendriform and q -tridendriform bialgebras, the Lie bracket comes from finer structures.

2.3. Definition. (see [4]) A *brace algebra* is a vector space B equipped with $n + 1$ -ary operations $M_{1n} : B \otimes B^{\otimes n} \rightarrow B$, for $n \geq 0$, which satisfy the following conditions:

- (1) $M_{10} = Id_B$,

$$\begin{aligned}
(2) \quad & M_{1m}(M_{1n}(x; y_1, \dots, y_n); z_1, \dots, z_m) = \\
& \sum_{0 \leq i_1 \leq j_1 \leq \dots \leq j_n \leq m} M_{1r}(x; z_1, \dots, z_{i_1}, M_{1l_1}(y_1; \dots, z_{j_1}), \dots, M_{1l_n}(y_n; \dots, z_{j_n}), \dots, z_m), \\
& \text{for } x, y_1, \dots, y_n, z_1, \dots, z_m \in B, \text{ where } l_k = j_k - i_k, \text{ for } 1 \leq k \leq n, \\
& \text{and } r = \sum_{k=1}^n i_k + m - j_n + n.
\end{aligned}$$

A *q*-Gerstenhaber-Voronov algebra, GV_q algebra for short, is a vector space B endowed with a brace structure given by operations M_{1n} and an associative product \cdot , satisfying the distributive relation:

$$\begin{aligned}
M_{1n}(x \cdot y; z_1, \dots, z_n) = \\
\sum_{0 \leq i \leq j \leq n} q^{j-i} M_{1i}(x; z_1, \dots, z_i) \cdot z_{i+1} \cdot \dots \cdot z_j \cdot M_{1(n-j)}(y; z_{j+1}, \dots, z_n),
\end{aligned}$$

for $x, y, z_1, \dots, z_n \in B$, where for $q = 0$ we fix that $q^j = 0$ if $j \geq 1$ and $q^0 = 1$.

Even if brace algebras and GV algebras have an infinite number of operations and seem much more complicated than dendriform and tridendriform algebras, the type of relations that these operations satisfy allow us to give an easy recursive formula for linear bases of the free objects of both theories.

Given a set X , let $\mathbb{M}(X) = \bigcup_{n \geq 1} \mathbb{M}_n(X)$ be subset of the free brace algebra $\text{Br}(X)$ over X , defined recursively by:

- (1) $M_0(X) := X$,
- (2) $M_1(X) := \{\mathbb{M}_{1m}(x; y_1, \dots, y_m) \mid x, y_1, \dots, y_m \in X, m \geq 1\}$,
- (3) $M_n(X) := \{M_{1m}(x; y_1, \dots, y_m) \mid x \in X \text{ and } y_i \in M_{j_i}(X) \text{ for } 1 \leq i \leq m, \text{ such that } \sum_{i=1}^m j_i = n - 1\}$.

For instance, for $x, y_1, y_2, y_3, y_4 \in X$,

- (1) $M_{14}(x; y_1, \dots, y_4) \in \mathbb{M}_1(X)$,
- (2) $M_{13}(x; y_1, M_{11}(y_2; y_3), y_4) \in \mathbb{M}_2(X)$,
- (3) $M_{12}(x; M_{11}(y_1; y_2), M_{11}(y_3; y_4)) \in \mathbb{M}_3(X)$.

The following Lemma is an immediate consequence of Definition 2.3.

2.4. Lemma. *For a set X , the set $M(X) := \bigcup_{n \geq 1} M_n(X)$ is a basis, as a vector space, of the free brace algebra $\text{Br}(X)$ spanned by X .*

In a similar way, for any set X , we define the subset $G(X) = \bigcup_{n \geq 1} G_n(X)$ of the free GV_q algebra $GV_q(X)$ over X , recursively as:

- (1) $G_0(X) := X$,

(2) $G_1(X) :=$

$\{y = M_{1m}(x; y_1, \dots, y_m) \text{ or } y = y_1 \cdot \dots \cdot y_m \mid x, y_1, \dots, y_m \in X, m \geq 1\},$

(3) For $n \geq 2$, the set $G_n(X)$ is the disjoint union of the subsets

$\{y = M_{1m}(x; y_1, \dots, y_m) \mid x \in X \text{ and } y_i \in G_{j_i}(X) \text{ for } 1 \leq i \leq m, \sum_{i=1}^m j_i = n-1\},$

and

$\{y = y_1 \cdot \dots \cdot y_m \mid y_i \in G_{k_i}(X) \text{ for } 1 \leq i \leq m, \sum_{i=1}^m k_i = n\}.$

As in the case of free brace algebras, Definition 2.3 implies the following result.

2.5. Lemma. *For a set X , the set $G(X) := \bigcup_{n \geq 1} G_n(X)$ is a basis, as a vector space, of the free GV_q algebra spanned by X , denoted $GV_q(X)$.*

2.6. Notation. Let (A, \prec, \succ) be a dendriform algebra. For a family of elements y_1, \dots, y_r in A , let $\omega^\prec(y_1, \dots, y_r)$ and $\omega^\succ(y_1, \dots, y_r)$ be the following elements of A :

- (1) $\omega^\prec(y_1, \dots, y_r) := y_1 \prec (y_2 \prec (\dots \prec (y_{r-1} \prec y_r))),$
- (2) $\omega^\succ(y_1, \dots, y_r) := (((y_1 \succ y_2) \succ y_3) \succ \dots) \succ y_r.$

There exists a functor from the category of dendriform algebras to the category of brace algebras.

2.7. Definition. Let (A, \succ, \prec) be a dendriform algebra. Define operations $M_{1n} : A^{\otimes(n+1)} \longrightarrow A$ as follows:

$$M_{1n}(x; y_1, \dots, y_n) := \sum_{r=0}^n (-1)^{n-i} \omega^\prec(y_1, \dots, y_r) \succ x \prec \omega^\succ(y_{r+1}, \dots, y_n),$$

for $n \geq 1$.

In [14], we proved that for any dendriform algebra (A, \succ, \prec) , the underlying vector space A with the $n+1$ -ary operations M_{1n} is a brace algebra. In the same work we showed the following result:

2.8. Proposition. *For any dendriform bialgebra H the subspace $\text{Prim}(H)$ is closed under the brace operations. The linear map $\varphi : T^c(\text{Prim}(H)) \longrightarrow H$, given by:*

$$\varphi(y_1 \otimes \dots \otimes y_r) := \omega^\succ(y_1, \dots, y_r),$$

for $y_1, \dots, y_r \in \text{Prim}(H)$ and $r \geq 1$, is a coalgebra epimorphism.

The following Theorem is proved in [14].

2.9. Theorem. *For any set X , the free dendriform algebra $\text{Dend}(X)$ over X has a natural structure of bialgebra. There exists a functor $\mathcal{U}_{\text{dend}}$ from the category $\text{Brace}_{\mathbb{K}}$ of brace algebras to the category of $\text{BiDend}_{\mathbb{K}}$ of dendriform bialgebras, left adjoint to Prim , satisfying that any conilpotent dendriform bialgebra H is isomorphic to $\mathcal{U}_{\text{dend}}(\text{Prim}(H))$.*

In [2], we proved that the functor from the category $\text{Dend}_{\mathbb{K}}$, of dendriform algebras over \mathbb{K} , into the category $\text{Brace}_{\mathbb{K}}$, which maps (A, \succ, \prec) into $(A, \{M_{1n}\}_{n \geq 1})$, composed with the functor from $\text{Tridend}_{q\mathbb{K}}$ to $\text{Dend}_{\mathbb{K}}$ factorizes through the category of GV_q algebras. That is $(A, \{M_{1n}\}_{n \geq 1}, \cdot)$ is a GV_q algebra, for all q -tridendriform algebra (A, \succ, \cdot, \prec) and we get

$$\begin{array}{ccc} \text{Tridend}_{q\mathbb{K}} & \rightarrow & \text{Dend}_{\mathbb{K}} \\ \downarrow & & \downarrow \\ \text{GV}_{q\mathbb{K}} & \rightarrow & \text{Brace}_{\mathbb{K}} \end{array}$$

Moreover, if $(H, \succ, \cdot, \prec, \Delta)$ is a q -tridendriform bialgebra, then $\text{Prim}(H)$ is closed under the brace operations M_{1n} and the associative product \cdot . Finally, we got the tridendriform version of Theorem 2.9:

2.10. Theorem. *Let X be a set, the free q -tridendriform algebra $\text{Tridend}_q(X)$ over X is isomorphic, as a coalgebra, to the cotensor coalgebra $T^c(\text{GV}_q(X))$, where $\text{GV}_q(X)$ denotes the free GV_q algebra spanned by X . The functor $\text{Prim} : \text{coBiTridend}_{q\mathbb{K}} \rightarrow \text{GV}_{q\mathbb{K}}$ is an equivalence of categories, where $\text{coBiTridend}_{q\mathbb{K}}$ denotes the category of conilpotent tridendriform bialgebras.*

3. PERMUTATIONS AND SURJECTIVE MAPS

We develop first some basic definitions and notations about surjective maps and shuffles.

For any positive integer $n \in \mathbb{N}$, let $[n]$ be the finite set $\{1, \dots, n\}$. We denote by S_n the set of permutations on $[n]$ and by \mathbf{ST}_n^r the set of surjective maps from $[n]$ to $[r]$. For $n \geq 1$, let $\mathbf{ST}_n := \bigcup_{r=1}^n \mathbf{ST}_n^r$.

Note that \mathbf{ST}_n^n coincides with the set S_n of permutations of n elements, while $\mathbf{ST}_n^1 = \{c_n\}$, where c_n is the constant function $c_n(i) = 1$, for $1 \leq i \leq n$.

For $f \in \mathbf{ST}_n^r$, we write $|f| = n$ and $f = (f(1), \dots, f(n))$. The composition of maps is denoted \circ .

The *concatenation product* $\times : \mathbf{ST}_n^r \times \mathbf{ST}_m^s \rightarrow \mathbf{ST}_{n+m}^{r+s}$ is given by the formula:

$$f \times g := (f(1), \dots, f(n), g(1) + r, \dots, g(m) + r).$$

3.1. Notation. We denote by 1_n the identity of S_n . For any pair of positive integers n and m , let $\epsilon(n, m)$ denote the permutation of $n + m$ elements whose image is $(m + 1, \dots, m + n, 1, \dots, m)$. For a finite collection of positive integers r_1, \dots, r_s with $s > 2$, the permutation $\epsilon(r_1, \dots, r_s)$ in $S_{r_1 + \dots + r_s}$ is the composition:

$$\epsilon(r_1, \dots, r_s) := \epsilon(r_1 + \dots + r_{s-1}, r_s) \circ (\epsilon(r_1, \dots, r_{s-1}) \times 1_{r_s}).$$

3.2. Definition. Given a map $f : [n] \rightarrow \mathbb{N}$ there exists a unique surjective map $\text{std}(f)$ in \mathbf{ST}_n^r such that $f(i) < f(j)$ if, and only if, $\text{std}(f)(i) < \text{std}(f)(j)$, for $1 \leq i, j \leq n$. The map $\text{std}(f)$ is called the *standardization* of f (see for instance [12]).

For example, when $f = (1, 5, 4, 7, 5)$, we get $\text{std}(f) = (1, 3, 2, 4, 3)$.

3.3. Notation. For $x \in \mathbf{ST}_n^r$ and $J = \{j_1 < \dots < j_k\} \subseteq \{1, \dots, n\}$, let $x|_J := \text{std}(x(j_1), \dots, x(j_k))$ denote the restriction of x to J .

Similarly, for $K = \{j_1 < \dots < j_l\} \subseteq \{1, \dots, r\}$, the co-restriction of x to K is denoted $x|_K := \text{std}(x(s_1), \dots, x(s_q))$, for $x^{-1}(K) = \{s_1 < \dots < s_q\}$.

For an element $x \in \mathbf{ST}_n^r$, we denote by $\lambda(x)$ the cardinal of $x^{-1}(\{r\})$.

Suppose that $x^{-1}(r) = \{j_1 < \dots < j_{\lambda(x)}\}$, and let $x' \in \mathbf{ST}_{n-k}^{r-1}$ be the co-restriction $x' := x|_{\{1, \dots, r-1\}}$. We denote x as $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$.

3.4. Example. For example, the surjective map $x = (3, 1, 2, 5, 1, 4, 3, 5, 4, 2)$ is written as $x = \prod_{4 < 8} (3, 1, 2, 1, 4, 3, 4, 2)$.

3.5. Definition. Let $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$ be an element of \mathbf{ST}_n^r , define the integer $\mathbb{M}(x)_i$, for $0 \leq i \leq \lambda(x)$ as follows:

$$\mathbb{M}(x)_i := \begin{cases} j_1 - 1, & \text{for } i = 0, \\ j_i - j_{i-1} - 1, & \text{for } 1 \leq i \leq \lambda(x) - 1, \\ n - j_{\lambda(x)}, & \text{for } i = \lambda(x). \end{cases}$$

We define $\mathbb{M}(x) := (\mathbb{M}(x)_{\lambda(x)}, \dots, \mathbb{M}(x)_1)$.

3.6. Definition. Let $x \in \mathbf{ST}_n^r$ be a surjective map and let $\underline{l} = (l_1, \dots, l_p)$ be a collection of integers such that $0 = l_0 < l_1 < \dots < l_p < n$. Define the element

$$x^{\underline{l}} := x|_{\{1, \dots, l_1\}} \times x|_{\{l_1+1, \dots, l_2\}} \times \dots \times x|_{\{l_{p-1}+1, \dots, n\}}.$$

For $p = 0$, define $x^{\underline{l}} = x$.

Recall that a *composition* of n is a collection (n_1, \dots, n_s) of positive integers such that $\sum n_i = n$.

3.7. Definition. Let (n_1, \dots, n_p) be a composition of n . An element in $f \in \mathbf{ST}_n$ is a (n_1, \dots, n_p) -*stuffle* if

$$f(n_1 + \dots + n_i + 1) < f(n_1 + \dots + n_i + 1) < \dots < f(n_1 + \dots + n_i + n_{i+1}),$$

for $0 \leq i \leq p-1$.

3.8. Notation. We denote by $SH(n_1, \dots, n_p)$ the set of all (n_1, \dots, n_p) -stuffles.

For a composition (n_1, \dots, n_p) of n , we denote:

- (1) $SH^<(n_1, \dots, n_p)$ the subset of all surjective maps $f \in SH(n_1, \dots, n_p)$ such that $f(n_1) > f(n_1 + n_2) > \dots > f(n)$.
- (2) $SH^>(n_1, \dots, n_p)$ the subset of all surjective maps $f \in SH(n_1, \dots, n_p)$ such that $f(n_1) < f(n_1 + n_2) < \dots < f(n)$.
- (3) $SH^\bullet(n_1, \dots, n_p)$ the subset of all surjective maps $f \in SH(n_1, \dots, n_p)$ such that $f(n_1) = f(n_1 + n_2) = \dots = f(n)$.
- (4) $SH^\geq(n_1, \dots, n_p)$ the subset of all surjective maps $f \in SH(n_1, \dots, n_p)$ such that $f(n_1) \leq f(n_1 + n_2) \leq \dots \leq f(n)$.

To recover the usual notion of shuffle, it suffices to note that a (n_1, \dots, n_p) -shuffle is a permutation $\sigma \in S_n \cap SH(n_1, \dots, n_p)$. We denote by $Sh(n_1, \dots, n_p)$ the set of all (n_1, \dots, n_p) -shuffles.

In an analogous way, we define $Sh^<(n_1, \dots, n_p) := S_n \cap SH^<(n_1, \dots, n_p)$, and $Sh^>(n_1, \dots, n_p) := S_n \cap SH^>(n_1, \dots, n_p)$.

Finally, we denote by $Sh^\bullet(r_1, \dots, r_p)$ the set of all $f = \prod_{r_1 < r_1 + r_2 < \dots < r_1 + \dots + r_p} f'$

with $f' \in Sh(r_1 - 1, \dots, r_p - 1)$.

The following property of the shuffles is well-known and is the key result to prove the associativity of the shuffle product.

3.9. Proposition. *Let n, m and r be positive integers. The set of (n, m, r) -shuffles satisfies the following property:*

$$Sh(n + m, r) \circ (Sh(n, m) \times 1_r) = Sh(n, m, r) = Sh(n, m + r) \circ (1_n \times Sh(m, r)),$$

where $1_n = (1, 2, \dots, n)$ denotes the identity of S_n .

3.10. Remark. A straightforward calculation shows that the equality of Proposition 3.9 splits into three formulas:

- (1) $Sh^>(n + m, r) \circ (Sh(n, m) \times 1_r) = Sh^>(n, m + r) \circ (1_n \times Sh^>(m, r))$,
- (2) $Sh^<(n + m, r) \circ (Sh^>(n, m) \times 1_r) = Sh^>(n, m + r) \circ (1_n \times Sh^<(m, r))$,
- (3) $Sh^<(n + m, r) \circ (Sh^<(n, m) \times 1_r) = Sh^<(n, m + r) \circ (1_n \times Sh(m, r))$.

The set of stuffles $SH(n, m, r)$ satisfies analogous properties. We shall use them to define tridendriform algebra structures, for details we refer to [15].

For $n \geq 1$ and $1 \leq i \leq n-1$, let $t_i \in S_n$ be the permutation which exchanges i and $i+1$, that is

$$t_i(j) := (1, \dots, i-1, i+1, i, i+2, \dots, n)$$

3.11. Definition. For $n \geq 1$, the weak Bruhat order on the set S_n of permutations is defined by the covering relation:

$$\sigma < t_i \circ \sigma,$$

when $\sigma^{-1}(i) < \sigma^{-1}(i+1)$.

The following Proposition is well-known, see for instance [1] or [10].

3.12. Proposition. For any composition (n_1, \dots, n_p) of n , the set of shuffles $Sh(n_1, \dots, n_p)$ coincides with the subset $\{w \in S_n \mid 1_n \leq w \leq \epsilon(n_1, \dots, n_p)\}$ of S_n , where \leq is the weak Bruhat order.

The weak Bruhat order of S_n may be extended to the set of surjective maps \mathbf{ST}_n in two different ways (see [15]). We describe the one we need in the last section of the paper.

3.13. Definition. For $n \geq 1$, the weak Bruhat order on \mathbf{ST}_n^r is the transitive relation spanned by the covering relation

$$f < t_i \circ f, \text{ when } f^{-1}(i) < f^{-1}(i+1),$$

for some $1 \leq i \leq r-1$, where for any pair of subsets $J, K \subseteq \{1, \dots, r\}$ we say that $J < K$ if the maximal element of J is smaller than the minimal element of K .

For instance, $(1, 4, 1, 3, 4, 2) < (2, 4, 2, 3, 4, 1)$, but the elements $(1, 4, 1, 3, 4, 2)$ and $(1, 3, 1, 4, 3, 2)$ are not comparable.

The following result is proved in [15].

3.14. Proposition. Let $\sigma < \tau \in Sh(r_1, \dots, r_p)$ be two permutations, and let $x_i \leq x'_i \in \mathbf{ST}_{n_i}^{r_i}$, for $1 \leq i \leq p-1$ be surjective maps. We have that:

- (1) $\sigma \circ (x_1 \times \dots \times x_p) < \tau \circ (x_1 \times \dots \times x_p)$,
- (2) $\sigma \circ (x_1 \times \dots \times x_p) \leq \sigma \circ (x'_1 \times \dots \times x'_p)$. Moreover, if at least for one $1 \leq i \leq p$ the elements x_i and x'_i satisfy that $x_i < x'_i$, then $\sigma \circ (x_1 \times \dots \times x_p) < \sigma \circ (x'_1 \times \dots \times x'_p)$.

4. THE COALGEBRA \mathcal{ST} OF SURJECTIVE MAPS

We define different algebraic structures on the graded vector space $\mathbb{K}[\mathbf{ST}] := \bigoplus_{n \geq 1} \mathbb{K}[\mathbf{ST}_n]$. We begin by the coassociative coproduct Δ . For a more detailed description of the properties of Δ see [12], [3] or [8].

4.1. Definition. We define $\Delta : \mathbb{K}[\mathbf{ST}] \longrightarrow \mathbb{K}[\mathbf{ST}] \otimes \mathbb{K}[\mathbf{ST}]$ on an element $x \in \mathbf{ST}_n^r$ by:

$$\Delta(x) = \sum_{i=1}^{r-1} x|^{1,\dots,i} \otimes x|^{i+1,\dots,r},$$

and we extend it by linearity to all $\mathbb{K}[\mathbf{ST}]$

For example,

$$\begin{aligned} \Delta(3, 4, 2, 5, 1, 1, 3, 5) &= (1, 1) \otimes (2, 3, 1, 4, 2, 4) + (2, 1, 1) \otimes (1, 2, 3, 1, 3) + \\ &\quad (3, 2, 1, 1, 3) \otimes (1, 2, 2) + (3, 4, 2, 1, 1, 3) \otimes (1, 1). \end{aligned}$$

For any $x \in \mathbf{ST}_n^r$ and any pair $1 < i < j < r - 1$, we have that:

- (1) $(x|^{1,\dots,j})|^{i+1,\dots,j} = x|^{i+1,\dots,j},$
- (2) $(x|^{i+1,\dots,r})|^{1,\dots,j} = x|^{i+1,\dots,j},$

which implies that the coproduct is coassociative.

Let \mathcal{ST} denote the graded coalgebra $(\mathbb{K}[\mathbf{ST}], \Delta)$. On \mathcal{ST}^+ , the coproduct Δ is uniquely extended to Δ^+ in such a way that the reduced coproduct of \mathcal{ST}^+ is Δ .

The data $(\mathcal{ST}^+, \Delta^+)$ is a coassociative unital and counital coalgebra.

It is clear that the concatenation product \times , extended by linearity, defines an associative graded product on \mathcal{ST} .

4.2. Definition. An element $f \in \mathbf{ST}_n^r$ is called *irreducible* if there do not exist an integer $1 \leq i \leq n - 1$ and a pair of surjective maps $g \in \mathbf{ST}_i^k$ and $h \in \mathbf{ST}_{n-i}^{r-k}$ such that $f = g \times h$. We denote by \mathbf{Irr}_n the set of irreducible elements of \mathbf{ST}_n , for $n \geq 1$, and by \mathbf{Irr} the union $\bigcup_{n \geq 1} \mathbf{Irr}_n$.

4.3. Remark. Given a surjective map $x \in \mathbf{ST}_n^r$, there exists a unique family x^1, \dots, x^p of elements, with $x^i \in \mathbf{Irr}_{n_i}^{r_i}$, such that $x = x^1 \times \dots \times x^p$, where $n = \sum_{i=1}^p n_i$ and $r = \sum_{i=1}^p r_i$. So, the space $\mathbb{K}[\mathbf{ST}]$ with \times is the free associative algebra spanned by the set \mathbf{Irr} .

4.4. Lemma. Let x and y be elements of \mathcal{ST}^+ , we have that:

$$\Delta^+(x \times y) = \sum (x_{(1)} \otimes (x_{(2)} \times y) + (x \times y_{(1)}) \otimes y_{(2)}) - x \otimes y,$$

where $\Delta^+(x) = \sum x_{(1)} \otimes x_{(2)}$, $\Delta^+(y) = y_{(1)} \otimes y_{(2)}$.

Proof. Suppose that $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$. To prove the Lemma it suffices to note that:

(1)

$$(x \times y)|^{1,\dots,i} = \begin{cases} x|^{1,\dots,i}, & \text{for } 0 \leq i \leq r, \\ x \times y|^{1,\dots,i-r}, & \text{for } r+1 \leq i \leq r+s, \end{cases}$$

(2)

$$(x \times y)|^{\{i+1, \dots, r+s\}} = \begin{cases} x|_{\{i+1, \dots, r\}} \times y, & \text{for } 0 \leq i \leq r, \\ y|_{\{i-r+1, \dots, s\}}, & \text{for } r+1 \leq i \leq r+s, \end{cases}$$

which imply the formula. \square

A vector space V equipped with an associative product and a coassociative coproduct, satisfying the condition of Lemma 4.4 is called an *infinitesimal unital bialgebra* in [11]. So, $(\mathcal{ST}^+, \times, \Delta^+)$ is a unital infinitesimal bialgebra.

As proved in [11], any conilpotent unital infinitesimal bialgebra (C, \times, Δ) is isomorphic, as a coalgebra, to the cotensor algebra $T^c(\text{Prim}(C))$. Moreover, the linear map

$$E(x) := \sum_{i \geq 1} (-1)^i \left(\sum x_{(1)} \times \dots \times x_{(i)} \right),$$

gives a projection from C to $\text{Prim}(C)$, where $\overline{\Delta}^i(x) = \sum x_{(1)} \otimes \dots \otimes x_{(i)}$ for $x \in C$. For the details of the construction we refer to [11].

4.5. Remark. For the particular case of $(\mathcal{ST}^+, \times, \Delta^+)$, we get that the linear map $E : \mathbb{K}[\mathbf{Irr}] \rightarrow \text{Prim}(\mathcal{ST}^+)$ is an isomorphism. It induces an isomorphism of coalgebras $E_{\mathcal{ST}} : T^c(\mathbb{K}[\mathbf{Irr}]) \rightarrow \mathcal{ST}$, given by:

$$E_{\mathcal{ST}}(x^1 \otimes \dots \otimes x^p) := E(x^1) \times \dots \times E(x^p),$$

for any family of irreducible elements $x^1, \dots, x^p \in \mathbf{Irr}$. On the other hand, we proved that $Id_{\mathcal{ST}_n} = \sum_{j=1}^n \times^j \circ E^{\otimes j} \circ \overline{\Delta}^j$, which implies that $T^c(\text{Prim}(\mathcal{ST}))$ is isomorphic to \mathcal{ST} , as coalgebras. Thus, putting the isos together, we have

$$T^c(\mathbb{K}[\mathbf{Irr}]) \cong \mathcal{ST} \cong T^c(\text{Prim}(\mathcal{ST})).$$

Noting that these isos respect the grading induced on the tensor algebras, we get that the dimension of the subspace $\text{Prim}(\mathcal{ST})_n$ of homogeneous elements of degree n is the cardinal $|\mathbf{Irr}_n|$ of the set of irreducible elements of degree n , for $n \geq 1$.

Using Definition 3.6, $E(x) = \sum_{\underline{l}} \alpha_{\underline{l}} x^{\underline{l}}$, where the sum is taken over all families $\underline{l} = (l_1, \dots, l_p)$ such that $0 = l_0 < l_1 < \dots < l_p < n$, $p \geq 0$ and $\alpha_{\underline{l}} := (-1)^p$, which implies that $x^{\underline{l}} = x$ or $x^{\underline{l}}$ is reducible.

4.6. Remark. If $x \in \mathbf{ST}_n^r$ and $\underline{l} = (l_1, \dots, l_p)$, then $x^{\underline{l}} \in \mathbf{ST}_n^r$, too.

4.7. Lemma. Let $x \in \mathbf{ST}_n^r$ and let $p \geq 1$. Given a family of integers \underline{l} such that $0 = l_0 < l_1 < l_2 < \dots < l_p < r$, the element $x^{\underline{l}}$ satisfies that $\mathbb{M}(x^{\underline{l}}) \leq \mathbb{M}(x)$ for the lexicographic order.

Proof. Suppose that $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$.

If $x^{-1}(\{1, \dots, l_p\}) \subseteq \{1, \dots, j_1 - 1\}$, then $x^L = \prod_{j_1 < \dots < j_{\lambda(x)}} (x')^L$, which implies that $\mathbb{M}(x^L) = \mathbb{M}(x)$.

On the other hand, if $x^{-1}(\{1, \dots, l_p\}) \cap \{j_1 + 1, \dots, n\} \neq \emptyset$, then there exists at least one $j_1 < k \leq n$ such that $x(k) \leq l_p$. Let k_0 be the maximal integer which satisfies this condition. There exists $1 \leq i_0 \leq p$ such that $j_{i_0} < k_0 < j_{i_0+1}$, and we get that:

- $\mathbb{M}(x^L)_i = \mathbb{M}(x)_i$ for $i_0 < i \leq \lambda(x)$,
- $\mathbb{M}(x^L)_{i_0} < \mathbb{M}(x)_{i_0}$.

So, $\mathbb{M}(x^L) < \mathbb{M}(x)$. □

5. THE DENDRIFORM BIALGEBRA \mathcal{ST}_D

The shuffle product defines a bialgebra structure on \mathcal{ST}^+ , which has been studied by F. Chapoton, and by J.-C. Novelli and J.-Y. Thibon, who called it the bialgebra of *packed words*. We recall the main constructions and results, for the details of the proofs we refer to [3] and [12].

Let $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$ be two surjective applications, the *shuffle product* $x * y$ is defined by:

$$x * y := \sum_{f \in Sh(r,s)} f \circ (x \times y).$$

The product $*$ is associative and satisfies that:

$$\Delta(x * y) = \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} * y_{(2)}),$$

for any $x, y \in \mathbf{ST}$. So, the coalgebra (\mathcal{ST}^+, Δ) equipped with the shuffle product $*$ is a bialgebra over \mathbb{K} .

Using Remark 3.10, the shuffle product of \mathcal{ST}^+ comes from a dendriform structure of \mathcal{ST} .

That is, the vector space \mathcal{ST} with the products \succ and \prec defined by:

- (1) $x \succ y := \sum_{f \in Sh^{\succ}(r,s)} f \circ (x \times y)$,
- (2) $x \prec y := \sum_{f \in Sh^{\prec}(r,s)} f \circ (x \times y)$,

for $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$, is a dendriform algebra.

Fix that $x \succ 1_{\mathbb{K}} := 0 =: 1_{\mathbb{K}} \prec x$ and $x \prec 1_{\mathbb{K}} := x =: 1_{\mathbb{K}} \succ x$, for all $x \in \mathcal{ST}$. Note that \mathcal{ST}^+ is not a dendriform algebra, because there does not exist a coherent way to define $1_{\mathbb{K}} \succ 1_{\mathbb{K}}$ and $1_{\mathbb{K}} \prec 1_{\mathbb{K}}$. It is easily seen that \mathcal{ST} is a conilpotent dendriform bialgebra.

The dendriform algebra $(\mathcal{ST}, \succ, \prec)$ is free. The main result of the present section is to give a proof of this result by exhibiting a basis.

5.1. Notation. Let $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$ be two maps, we denote by $x \setminus y$ the composition:

$$x \setminus y := \epsilon(r, s) \circ (x \times y) \in \mathbf{ST}_{n+m}^{r+s}.$$

For $n \geq 1$, define the subset $\mathcal{D}_n \subseteq \mathbf{ST}_n$ recursively, as follows:

- (1) $\mathcal{D}_1 := \{(1)\} = \mathbf{ST}_1$,
- (2) $\mathcal{D}_2 := \{(1, 1)\}$,
- (3) a surjective map $x \in \mathbf{ST}_n$ belongs to \mathcal{D}_n if x is irreducible, and there do not exist an integer $1 \leq r < n$ and a pair of elements $y \in \mathcal{D}_r$ and $z \in \mathbf{ST}_{n-r}$ such that $x = y \setminus z$.

5.2. Theorem. *The dendriform algebra $(\mathcal{ST}, \succ, \prec)$ is the free dendriform algebra spanned by the set*

$$E(\mathcal{D}) = \{E(x) \mid x \in \bigcup_{n \geq 1} \mathcal{D}_n\},$$

where $E(x) = \sum_{i \geq 1} (-1)^i (\sum x_{(1)} \times \dots \times x_{(i)})$, with $\overline{\Delta}^i(x) = \sum x_{(1)} \otimes \dots \otimes x_{(i)}$, for $i \geq 1$.

In order to prove Theorem 5.2, we need some additional results. Note first that:

5.3. Remark. For any irreducible element $x \in \mathbf{Irr}$, there exist a unique integer $1 \leq m \leq n$ and unique elements $y \in \mathcal{D}_m$ and $z \in \mathbf{ST}_{n-m}$ such that $x = y \setminus z$. Moreover, there exists a unique way to write down $z = z^1 \times \dots \times z^p$, with $z^i \in \mathbf{Irr}$, for $1 \leq i \leq p$.

The next Lemmas will serve in the proof of Theorem 5.2.

5.4. Lemma. *Let $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$. For any $f \in Sh(r, s)$, we have that:*

- (1) $f \circ (x \times y)|_{\{1, \dots, n\}} = x$,
- (2) $f \circ (x \times y)|_{\{n+1, \dots, n+m\}} = y$.

Proof. The result is an easy consequence of

$$f(1) < \dots < f(r), \text{ and } f(r+1) < \dots < f(r+s).$$

□

5.5. Notation. We denote by $\mathfrak{B}(l)$ the set of all the elements x in \mathbf{ST} which are of the form $x = y \setminus z$ with $y \in \mathcal{D}$ and $|z| = l$, for $l \geq 0$. Note that $\mathfrak{B}(0) = \mathcal{D}$.

5.6. Lemma. *Let $x \in \mathcal{D}_n^r$ and $y \in \mathbf{ST}_m^s$ be two surjective maps and let $f \in Sh^{\prec}(r, s)$, $f \neq \epsilon(r, s)$. If there exist $z \in \mathcal{D}$ and $w \in \mathbf{ST}$ such that $f \circ (x \times y) = z \setminus w \in \mathfrak{B}(l)$, then $l < m$.*

Proof. Suppose that $f \circ (x \times y) = z \setminus w$ with $|w| \geq m$. In this case, $|z| \leq n$, and from Lemma 5.4 we get that:

$$x = f \circ (x \times y)|_{\{1, \dots, n\}} = z \setminus (w|_{\{1, \dots, n-|z|\}}).$$

As $x \in \mathcal{D}$, then $x = z$ and $n = |z|$. So, we get $f \circ (x \times y) = x \setminus w$, but this is possible only when $f = \epsilon(r, s)$. \square

5.7. Lemma. *Let x be a reducible element of \mathbf{ST}_n^r and let $y \in \mathbf{ST}_m^s$. For any $f \in \text{Sh}^{\prec}(r, s)$, we have that either $f \circ (x \times y)$ is reducible, or $f \circ (x \times y) \in \mathfrak{B}(l)$, for $l < m$.*

Proof. Suppose that $f \circ (x \times y)$ is irreducible. In this case, there exist $z \in \mathcal{D}$ and $w \in \mathbf{ST}$ such that $f \circ (x \times y) = z \setminus w$.

Lemma 5.4 states that $x = f \circ (x \times y)|_{\{1, \dots, n\}} = (z \setminus w)|_{\{1, \dots, n\}}$. If $|z| \leq n$, we get that z is reducible, which is false. So, $|z| > n$, which implies that $f \circ (x \times y) = z \setminus w \in \mathfrak{B}(l)$, for $l < m$. \square

Define, on the vector space $\mathbb{K}[\mathbf{Irr}]$, a structure of brace algebra given by:

- (1) for $x \in \mathcal{D}$ and $y_1, \dots, y_n \in \mathbf{Irr}$,

$$M_{1n}(x; y_1, \dots, y_n) := x \setminus (y_1 \times \dots \times y_n),$$

- (2) for $x \in \mathbf{Irr} \setminus \mathcal{D}$ and $y_1, \dots, y_n \in \mathbf{Irr}$, there exist $x_1 \in \mathcal{D}$ and $x_2 \in \mathbf{ST}$ such that $x_2 = z_1 \times \dots \times z_p$ with $z_j \in \mathbf{Irr}$, for $1 \leq j \leq p$.

In this case, $M_{1n}(x; y_1, \dots, y_n) = M_{1n}(M_{1p}(x_1; z_1, \dots, z_p); y_1, \dots, y_n)$ is defined using Definition 2.3.

Lemma 2.4 states that $(\mathbb{K}[\mathbf{Irr}], \{M_{1n}\}_{n \geq 1})$ is a well-defined brace algebra.

A recursive argument on $|y|$, for $y \in \mathbf{Irr}$, shows that as a brace algebra $\mathbb{K}[\mathbf{Irr}]$ is freely generated by \mathcal{D} .

Let $E|_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Prim}(\mathcal{ST})$ be the restriction to \mathcal{D} of the projection $E(x) = \sum_{i \geq 1} (-1)^i (\sum x_{(1)} \times \dots \times x_{(i)})$. As $\text{Prim}(\mathcal{ST})$ is a brace algebra, there exists a unique homomorphism of brace algebras $\eta : \mathbb{K}[\mathbf{Irr}] \rightarrow \text{Prim}(\mathcal{ST})$, such that $\eta(x) = E(x)$, for $x \in \mathcal{D}$.

We want to prove that η is an isomorphism.

5.8. Notation. Let $x \in \mathbf{Irr}$, we denote $\eta(x) = \sum_{a_i \neq 0} a_i x_i$. There exists a unique i_0 such that $x_{i_0} = x$ and $a_{i_0} = 1$.

Given an element $x = y \setminus z \in \mathbf{Irr}_n$, with $y \in \mathcal{D}_m$ and $z = z^1 \times \dots \times z^p$, with $z^i \in \mathbf{Irr}$ for $1 \leq i \leq p$, the homomorphism η is defined as

$$\begin{aligned} \eta(x) &= M_{1p}^{ST}(\eta(y); \eta(z^1), \dots, \eta(z^p)) = \\ &\sum_j (-1)^j \left(\sum c_{k, i_1, \dots, i_p} \omega^{\prec}(z_{i_1}^1, \dots, z_{i_j}^j) \succ y^{\perp} \prec \omega^{\succ}(z_{i_{j+1}}^{j+1}, \dots, z_{i_p}^p) \right), \end{aligned}$$

where M_{1p}^{ST} denote the brace operations in \mathcal{ST} and $\eta(z^j) = \sum_{b_{i_j} \neq 0} b_{i_j} z_{i_j}^j$, for $1 \leq j \leq p$.

5.9. Lemma. *Let $x = y \setminus z$ be an element in $\mathfrak{B}(l)$, with $y \in \mathcal{D}_n^r$. If $\eta(x) = \sum_{a_i \neq 0} a_i x_i$, then:*

- (1) $\mathbb{M}(x_i) \leq \mathbb{M}(x)$ for the lexicographic order,
- (2) if $\mathbb{M}(x_i) = \mathbb{M}(x)$, then x_i is reducible or $x_i \in \mathfrak{B}(k)$, with $0 \leq k \leq l$,

for all i .

Suppose that $z = z^1 \times \dots \times z^p$, with $z^j \in \mathbf{Irr}_{m_j}^{s_j}$ and $\eta(z^j) = \sum_{b_{i_j} \neq 0} b_{i_j} z_{i_j}^j$. The unique terms x_i satisfying that $\mathbb{M}(x_i) = \mathbb{M}(x)$ and $x_i \in \mathfrak{B}(l)$, are of the form:

$$x_i = y \setminus (g \circ (z_{i_1}^1, \dots, z_{i_p}^p)),$$

with $g \in \text{Sh}^\succ(s_1, \dots, s_p)$, for some family $z_{i_j}^j$.

Proof. Note first that $\lambda(x) = \lambda(y)$ and $\mathbb{M}(x)_{\lambda(x)} = \mathbb{M}(y)_{\lambda(y)} + |z|$.

As $y \in \mathcal{D}$, we have that $\eta(y) = \sum_l \alpha_l y^l$, with $\mathbb{M}(y^l) \leq \mathbb{M}(y)$ and y^l reducible or $y^l = y$, by Lemma 4.7.

Recall that

$$\omega^\prec(z_{i_1}^1, \dots, z_{i_j}^j) \succ y^l \prec \omega^\succ(z_{i_{j+1}}^{j+1}, \dots, z_{i_p}^p) = \sum_{f,g,h} h \circ (f \circ (z_{i_1}^1 \times \dots \times z_{i_j}^j) \times y^l \times g \circ (z_{i_{j+1}}^{j+1} \times \dots \times z_{i_p}^p)),$$

where $f \in \text{Sh}^\prec(s_1, \dots, s_j)$, $g \in \text{Sh}^\succ(s_{j+1}, \dots, s_p)$ and $h \in \text{Sh}(s_1 + \dots + s_j, r, s_{j+1} + \dots + s_p)$ is such that $h(s_1 + \dots + s_j) < h(s_1 + \dots + s_j + r)$ and $h(s_1 + \dots + s_j + r) > h(s_1 + \dots + s_p + r)$.

Let $x_i := h \circ (f \circ (z_{i_1}^1 \times \dots \times z_{i_j}^j) \times y^l \times g \circ (z_{i_{j+1}}^{j+1} \times \dots \times z_{i_p}^p))$.

We have that $\lambda(x_i) = \lambda(y^l)$ and $\mathbb{M}(x_i)_{\lambda(x_i)} = \mathbb{M}(y^l)_{\lambda(y^l)} + |z^{j+1}| + \dots + |z^p|$. So,

- (1) $\mathbb{M}(x_i) \leq \mathbb{M}(x)$, for all x_i ,
- (2) if $j \geq 1$, then $\mathbb{M}(x_i) < \mathbb{M}(x)$,
- (3) if $\mathbb{M}(y^l) < \mathbb{M}(y)$, then $\mathbb{M}(x_i) < \mathbb{M}(x)$.

The unique elements such that $\mathbb{M}(x_i) = \mathbb{M}(x)$ are of the form

$$x_i = h \circ (y^l \times g \circ (z_{i_1}^1, \dots, z_{i_p}^p)),$$

where $\mathbb{M}(y^l) = \mathbb{M}(y)$, $g \in \text{Sh}^\succ(s_1, \dots, s_p)$ and $h \in \text{Sh}^\prec(r, s_1 + \dots + s_p)$.

But, if $y^l \neq y$, then y^l is reducible, and by Lemma 5.7, we have that either x_i is reducible or $x_i \in \mathfrak{B}(k)$, with $0 \leq k < l$. So, we may restrict ourselves to consider only the x_i of the form:

$$x_i = h \circ (y \times g \circ (z_{i_1}^1, \dots, z_{i_p}^p)),$$

where $g \in \text{Sh}^\succ(s_1, \dots, s_p)$ and $h \in \text{Sh}^\prec(r, s_1 + \dots + s_p)$.

Lemma 5.6 implies that for $h \neq \epsilon(r, s_1 + \dots + s_p)$, the element x_i belongs to $\mathfrak{B}(k)$, for $k < l$.

The unique terms such that $\mathbb{M}(x_i) = \mathbb{M}(x)$ and $x_i \in \mathfrak{B}(l)$, are of the form:

$$x_i = y \setminus (g \circ (z_{i_1}^1, \dots, z_{i_p}^p)),$$

where $g \in \text{Sh}^\succ(s_1, \dots, s_p)$. □

5.10. Definition. For $x \in \mathbf{Irr}$, define $\overline{\eta}(x)$ as follows:

- (1) $\overline{\eta}(x) := x$, for $x \in \mathcal{D}$,
- (2) $\overline{\eta}(x) := y \setminus \omega^\succ(\eta(z^1), \dots, \eta(z^p))$,
for $x = y \setminus z \in \mathfrak{B}(l)$, with $y \in \mathcal{D}$ and $z = z^1 \times \dots \times z^p$, such that $z^j \in \mathbf{Irr}$.

5.11. Corollary. *If the set $\{\overline{\eta}(x) \mid x \in \mathbf{Irr}\}$ is linearly independent in \mathcal{ST} , then the set $\{\eta(x) \mid x \in \mathbf{Irr}\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$.*

Proof of Theorem 5.2

Recall from Remark 4.5 that $\dim_{\mathbb{K}}(\mathbb{K}[\mathbf{Irr}_n])$ and $\dim_{\mathbb{K}}(\text{Prim}(\mathcal{ST})_n)$ are equal. Hence it suffices to verify that η is either injective or surjective, grade by grade.

For any irreducible element x , we have that $E(x) = \sum_{\underline{l}} \alpha_{\underline{l}} x^{\underline{l}}$, where $x^{\underline{l}}$ is reducible for all $\underline{l} = (l_1, \dots, l_p)$, $p \geq 1$. So, the set $\{\eta(x) \mid x \in \mathcal{D}\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$.

Applying Corollary 5.11, it suffices to show that the set $\{\overline{\eta}(x) \mid x \in \mathbf{Irr}\}$ is linearly independent in \mathcal{ST} .

As $\text{Prim}(\mathcal{ST}) = \bigoplus_{n \geq 1} \text{Prim}(\mathcal{ST})_n$, we prove the result by induction on n .

For $n = 1$, $\text{Prim}(\mathcal{ST})_1 = \mathbb{K} \cdot (1)$ and $(1) = \eta(1) = \overline{\eta}(1)$.

For $n = 2$, $\mathbf{Irr}_2 = \{(2, 1); (1, 1)\}$. We have that $\overline{\eta}((2, 1)) = (2, 1)$ and $\overline{\eta}(1, 1) = (1, 1)$, which proves the result.

For $n \geq 3$, suppose that $\overline{\eta}(\mathbf{Irr}_m)$ is linearly independent for all $m < n$. We have that $\eta(\mathbf{Irr}_m)$ is linearly independent in $\text{Prim}(\mathcal{ST})$ for all $m < n$,

which implies that $\bigoplus_{j=1}^{n-1} \text{Prim}(\mathcal{ST})_j$ is spanned by $\bigcup_{j=1}^{n-1} \eta(\mathbf{Irr}_j)$.

Consider the linear map $\overline{\varphi} : \mathcal{ST} \rightarrow \mathcal{ST}$ defined by:

$$\overline{\varphi}(z) := \omega^\succ(\eta(z^1), \dots, \eta(z^p)),$$

for $z = z^1 \times \dots \times z^p$ with z^1, \dots, z^p irreducible elements.

Proposition 2.8 asserts that any element of $\bigoplus_{j=1}^{n-1} \mathcal{ST}_j$ belongs to

$$\overline{\varphi}(\bigoplus_{j=1}^{n-1} \text{Prim}(\mathcal{ST}_j)) = \overline{\varphi}(\mathbb{K}[\bigcup_{j=1}^{n-1} \eta(\mathbf{Irr}_j)]).$$

Therefore the set $\{\overline{\varphi}(z) \mid z \in \bigcup_{j=1}^{n-1} \mathbf{ST}_j\}$ spans $\bigoplus_{j=1}^{n-1} \mathcal{ST}_j$, which implies that it is linearly independent.

For any $y \in \mathcal{D}$ fixed, we get that $\{y \backslash \overline{\varphi}(z) \mid z \in \bigcup_{j=1}^{n-1} \mathbf{ST}_j\}$ is linearly independent in \mathcal{ST} .

But, for any $x \in \mathbf{Irr}$ there exist unique elements $y \in \mathcal{D}$ and $z \in \mathbf{ST}$ such that $x = y \backslash z$, so we may conclude that

$$\{\overline{\eta}(x) \mid x \in \mathbf{Irr}\} = \{y \backslash \overline{\varphi}(z) \mid y \in \mathcal{D} \text{ and } z \in \mathbf{ST}\}$$

is linearly independent, which ends the proof.

6. TRIDENDRIFORM ALGEBRA STRUCTURES ON \mathcal{ST}

We denote \mathcal{ST}_{qT} the q -tridendriform algebra, whose underlying vector space is $\mathbb{K}[\mathbf{ST}]$, which is described in this section.

The 0-tridendriform algebra \mathcal{ST}_{0T} was introduced by F. Chapoton in [3], while the 1-tridendriform structure \mathcal{ST}_{1T} was described in [8].

For $f \in \mathbf{ST}_n^r$, we denote by $s(f)$ the integer $n-r$. Using the conventions of Notation 3.8, we define a q -tridendriform structure on $\mathbb{K}[\mathbf{ST}]$ in terms of stuffles.

6.1. Definition. The binary operations \succ_q , \cdot_q and \prec_q are defined on the vector space $\mathbb{K}[\mathbf{ST}]$ as follows:

- (1) $x \succ_q y := \sum_{f \in SH^{\succ}(r,s)} q^{s(f)} f \circ (x \times y),$
- (2) $x \cdot_q y := \sum_{f \in SH^{\bullet}(r,s)} q^{s(f)-1} f \circ (x \times y),$
- (3) $x \prec_q y := \sum_{f \in SH^{\prec}(r,s)} q^{s(f)} f \circ (x \times y),$

for $x \in \mathbf{ST}_n^r$ and $y \in \mathbf{ST}_m^s$, where when $q = 0$ we establish that $q^0 := 1$.

For example, if $f = (2, 1, 1) \in \mathbf{ST}_3$ and $g = (1, 2) \in \mathbf{ST}_2$, then

$$f \succ_q g = (2, 1, 1, 3, 4) + q(2, 1, 1, 1, 3) + q(2, 1, 1, 2, 3) + (3, 1, 1, 2, 4) + (3, 2, 2, 1, 4),$$

$$f \cdot_q g = q(2, 1, 1, 1, 2) + (3, 2, 2, 1, 3) + (3, 1, 1, 2, 3),$$

$$f \prec_q g = q(3, 1, 1, 1, 2) + q(3, 2, 2, 1, 2) + (4, 1, 1, 2, 3) + (4, 2, 2, 1, 3) + (4, 3, 3, 1, 2).$$

6.2. Remark. When $q = 0$, the definition of \succ_0 , \prec_0 and \cdot_0 are simpler than the general case. For instance, we get that

$$x \succ_0 y := \sum_{f \in \text{Sh}^\succ(r, s)} f \circ (x \times y),$$

and a similar formula for \prec_0 . In the case of \cdot_0 , the sum described in Definition 6.1 is taken over all $f \in \text{Sh}^\bullet(r, s)$.

The following result was proved in [2], we refer to it for the details of the proof.

6.3. Proposition. *For any $q \in \mathbb{K}$, the space $\mathbb{K}[\mathbf{ST}]$ with the operations \succ_q , \cdot_q and \prec_q is a q -tridendriform algebra.*

6.4. Notation. The q -tridendriform algebra $(\mathbb{K}[\mathbf{ST}], \succ_q, \cdot_q, \prec_q)$ described in Definition 6.1 is denoted \mathcal{ST}_{qT} .

The following result is also proved in [2].

6.5. Proposition. *The q -tridendriform algebra \mathcal{ST}_{qT} with the coproduct Δ , described in Definition 4.1, is a conilpotent q -tridendriform bialgebra.*

Note that the underlying coalgebra structure of \mathcal{ST}_{qT} is \mathcal{ST} , so the subspace of primitive elements of \mathcal{ST}_{qT} is $\text{Prim}(\mathcal{ST})$.

7. $\mathbb{K}[\mathbf{Irr}]$ AS A FREE GV_q ALGEBRA

Our final goal is to exhibit a basis of \mathcal{ST}_{qT} as a free q -tridendriform algebra. The outline of the proof is similar to the one we used to construct a basis of \mathcal{ST}_D as a free dendriform algebra.

In the present section we show that the graded vector space $\mathbb{K}[\mathbf{Irr}]$ admits a structure of free GV_q algebra and in the last section we prove that there exists an isomorphism of GV_q algebras from $\mathbb{K}[\mathbf{Irr}]$ to $\text{Prim}(\mathcal{ST})$.

7.1. Definition. Let $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x' \in \mathbf{ST}_n^r$ and $y = \prod_{k_1 < \dots < k_{\lambda(y)}} y'$ be two surjective maps. Define the product

$$x \cdot y = \prod_{j_1 < \dots < j_{\lambda(x)} < k_1 + r < \dots < k_{\lambda(y)} + r} x' \times y'.$$

It is easy to verify that \cdot is associative.

We begin by constructing our basis.

Let \mathcal{C}_n be the set of irreducible elements of \mathbf{ST}_n defined recursively as follows:

- (1) $\mathcal{C}_1 = \emptyset$.

- (2) An element $x \in \mathbf{Irr}_n^r$ belongs to \mathcal{C}_n if it fulfills one of the following conditions:

- i. there exist $y \in \mathbf{Irr}_m^s \setminus \mathcal{C}_m$ and $z \in \mathbf{ST}_{r-s}^{n-m}$ such that $x = y \setminus z$,
- ii. there exists $x^1 \in \mathbf{ST}_m^s$ and $x^2 \in \mathbf{ST}_{n-m}^{r-s+1}$, such that $x = x^1 \cdot x^2$.

Note that, if $x = (r, x(2), \dots, x(n)) \in \mathbf{ST}_n^r$, then $x = (1) \cdot (x|_{\{2, \dots, n\}}) \in \mathcal{C}_n$. And if $x = (x(1), \dots, x(n-1), r) \in \mathbf{ST}_n^r$, then $x = x|_{\{1, \dots, n-1\}} \cdot (1) \in \mathcal{C}_n$.

- 7.2. Example.** (1) The element $x = (2, 5, 1, 3, 5, 2, 4, 5, 4)$ belongs to \mathcal{C}_9 because it fulfills the third condition. Indeed, $x^1 = (2, 4, 1, 3, 2) \in \mathbf{ST}_5^4$, $x^2 = (1, 2, 1) \in \mathbf{ST}_3^2$, and $x = x^1 \cdot x^2$.
- (2) The element $y = (4, 5, 2, 3, 1)$ belongs to \mathcal{C}_5 as it verifies the second condition, for $x^1 = (3, 4, 1, 2)$ and $x^2 = (1)$. Note that $x \notin \mathcal{C}_4$.

7.3. Definition. An *indecomposable map* is an element $x \in \mathbf{ST}_n^r$ such that there do not exist surjective maps x^1 and x^2 satisfying that $x = x^1 \cdot x^2$. We denote by **Indec** the set of indecomposable elements of \mathbf{ST} .

A standard argument shows that for any $x \in \mathbf{ST}$ there exist a unique integer $p \geq 1$ and unique indecomposable elements x^1, \dots, x^p such that x^2, \dots, x^p are irreducible and $x = x^1 \cdot \dots \cdot x^p$.

For instance, the element $x = (2, 3, 7, 1, 3, 4, 7, 5, 6, 7, 5)$ may be written as $x = (2, 3, 5, 1, 3, 4) \cdot (1, 2, 3, 1)$, as $(2, 3, 7, 1, 3) \cdot (1, 4, 2, 3, 4, 2)$ or as $(2, 3, 7, 1, 3) \cdot (1, 2) \cdot (1, 2, 3, 1)$, but the unique decomposition with x^2 irreducible is the first one.

Let \mathcal{B}_n be the set $\mathbf{Irr}_n \setminus \mathcal{C}_n$, for $n \geq 1$.

7.4. Definition. We have already introduced the product \cdot on $\mathbb{K}[\mathbf{Irr}]$. Define the operations M_{1n} as follows:

- (1) $M_{1n}(x; y_1, \dots, y_n) := x \setminus (y_1 \times \dots \times y_n)$, for any $x \in \bigcup_{n \geq 1} \mathcal{B}_n$ and any family of irreducible functions y_1, \dots, y_n .
- (2) for an $x \in \mathcal{C}_n$, we have two possibilities:
 - (a) If x is indecomposable, then $x \in \mathcal{B}_n$ or $x = y \setminus z$, with $y \in \mathcal{B}_m$ and $z \in \mathbf{ST}_{n-m}$.
For $x \notin \mathcal{B}_n$, there exist unique irreducible functions z^1, \dots, z^p such that $z = z^1 \times \dots \times z^p$, and $x = M_{1p}(y; z^1, \dots, z^p)$.
So, the element

$$M_{1n}(x; w^1, \dots, w^n) = M_{1n}(M_{1p}(y; z^1, \dots, z^p); w^1, \dots, w^n)$$

is well defined applying Definition 2.3 and Lemma 2.5.

- (b) If $x = x^1 \cdot \dots \cdot x^p$, for some $x^i \in \mathbf{ST}_{n_i}$ and some $p \geq 2$, then, for each $1 \leq i \leq p$, we may suppose that either $x^i \in \mathcal{B}_{n_i}$ or $x^i = y^i \setminus z^i$, with $y^i \in \mathcal{B}_{s_i}$. Again, applying the definition of \mathbf{GV}_q algebra, $M_{1n}(x; w_1, \dots, w_n)$ may be computed in terms of products of elements of type $M_{1l_j}(x^j; w_{i_1}, \dots, w_{i_j})$ and w_j .

So, we have a natural structure of GV_q algebra on $\mathbb{K}[\mathbf{Irr}]$.

7.5. Proposition. *The algebra $\mathbb{K}[\mathbf{Irr}]$, with the structure described in Definition 7.4, is the free GV_q algebra spanned by $\bigcup_{n \geq 1} \mathcal{B}_n$.*

Proof. Any element $x \in \mathbf{Irr}$ is written uniquely as a product $x = x^1 \cdot \dots \cdot x^p$, for $p \geq 1$, with $x^i \in \bigcup_{n \geq 1} \mathcal{B}_n$ or $x^i = M_{1n_i}(y^i; z_1^i, \dots, z_{n_i}^i)$, for all $1 \leq i \leq p$. So, $\bigcup_{n \geq 1} \mathcal{B}_n$ spans $\mathbb{K}[\mathbf{Irr}]$ as a free GV_q algebra. \square

8. FREEDOM OF \mathcal{ST}_{qT} AS A q -TRIDENDRIFORM ALGEBRA

Using Theorem 2.10 and arguments similar to those in section 5, we shall prove that $\text{Prim}(\mathcal{ST})$ is the free GV_q algebra spanned by \mathcal{B} , which implies that \mathcal{ST}_{qT} is the free q -tridendriform algebra spanned by \mathcal{B} .

Again, define $\psi^q : \mathbb{K}[\mathbf{Irr}] \rightarrow \text{Prim}(\mathcal{ST})$ as the homomorphism of GV_q algebras defined by setting:

$$\psi^q(x) := E(x) = \sum_{\underline{l}} \alpha_{\underline{l}} x^{\underline{l}},$$

for $x \in \mathcal{B}$.

8.1. Theorem. *The homomorphism $\psi^q : \mathbb{K}[\mathbf{Irr}] \rightarrow \text{Prim}(\mathcal{ST}_{qT})$ is an isomorphism, for all $q \in \mathbb{K}$.*

The rest of this section is devoted to the proof of Theorem 8.1, which implies that $\text{Prim}(\mathcal{ST})$ is a free GV_q algebra, and therefore that \mathcal{ST}_{qT} is a free q -tridendriform algebra.

As $|\mathbf{Irr}_n| = \dim_{\mathbb{K}}(\text{Prim}(\mathcal{ST})_n)$, for $n \geq 1$, it suffices to see that $\psi^q|_{\mathbb{K}[\mathbf{Irr}]_n}$ is surjective, or injective, for all $n \geq 1$.

Consider the following subsets of the set \mathbf{Irr}_{ST} :

- \mathcal{B} is the basis.
- \mathbf{Br} , is the set of elements of the form $x = y \setminus z$, with $y \in \mathcal{B}$, and $z \in \mathbf{ST}$. The set \mathbf{Br} is the disjoint union

$$\mathbf{Br} = \bigcup_{n \geq 0} \mathbf{Br}(n),$$

where $\mathbf{Br}(n)$ the subset of elements such that $|z| = n$. For $n = 0$, we have $\mathbf{Br}(0) = \mathcal{B}$.

- **Prod**, is the set of decomposable elements of the form $x = x^1 \cdot x^2$, with x^1 and x^2 irreducible. Define

$$\mathbf{Prod}(p) := \{x \in \mathbf{Prod} \mid \text{such that } x = x^1 \cdot \dots \cdot x^p, \text{ with } x^i \in \bigcup_{l \geq 0} \mathbf{Br}(l)\},$$

so that **Prod** is the disjoint union of $\{\mathbf{Prod}(p)\}_{p \geq 2}$.

Note that, by the definition of \mathcal{B} , we get that:

- (1) **Irr** is the union of **Br** and **Prod**,
- (2) $\mathcal{B} \cap \bigcup_{m \geq 1} \mathbf{Br}(m) = \emptyset$ and $\mathcal{B} \cap \mathbf{Prod} = \emptyset$.

Any element $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$ in $\bigcup_{m \geq 1} \mathbf{Br}(m)_n$ satisfies that $x^{-1}(\{1\}) \subseteq \{j_{\lambda(x)} + 1, \dots, n\}$.

On the other hand, if $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x' \in \mathbf{Prod}$ we have that $x^{-1}(\{1\}) \subseteq \{1, \dots, j_{\lambda(x)} - 1\}$, which implies that $\mathbf{Br} \cap \mathbf{Prod} = \emptyset$.

Let us recall the definition of $\psi^q(x)$ for $x \in \bigcup_{l \geq 1} \mathbf{Br}(l)$ and $x \in \mathbf{Prod}$. We

denote by $\cdot^{ST(q)}$ and $M_{1n}^{ST(q)}$ the associative product and the brace operations defined on \mathcal{ST}_{qT} .

8.2. Notation. As in section 5, for $x \in \mathbf{Irr}$, we denote $\psi^q(x) = \sum_{a_i \neq 0} a_i x_i$.

For $x \in \mathcal{B}_n^r$ and $\underline{l} = (l_1, \dots, l_p)$, the element $x^{\underline{l}}$ belongs to \mathbf{ST}_n^r , which implies that $\psi^q(x) \in \mathbb{K}[\mathbf{ST}_n^r]$.

For example,

- $\psi((2, 3, 1)) = (2, 3, 1) - (2, 1, 3)$, $\psi((1, 2, 1)) = (1, 2, 1) - (1, 1, 2)$,
- $\psi((2, 3, 4, 1)) = (2, 3, 4, 1) - (2, 3, 1, 4)$,
- $\psi((2, 4, 3, 1)) = (2, 4, 3, 1) - (2, 1, 4, 3) + (2, 1, 3, 4) - (2, 3, 1, 4)$.

Using Notation 3.8, we may describe easily $\psi(x)$, for $x \in \bigcup_{l \geq 1} \mathbf{Br}(l)$.

8.3. Notation. Let x^1, \dots, x^p be a collection of irreducible elements in \mathbf{ST} , with $x^j \in \mathbf{ST}_{n_j}^{r_j}$. For $q \in K$, we denote by:

- (1) $\gamma^{\succsim q}(x^1, \dots, x^p)$ the element:

$$(\dots ((x^1 \succsim_q x^2) \succsim_q \dots) \succsim_q x^p = \sum_{f \in SH^{\succsim}(r_1, \dots, r_p)} q^{s(f)} f \circ (x^1 \times \dots \times x^p),$$

- (2) $\gamma^{\succ q}(x^1, \dots, x^p)$ the element:

$$(\dots ((x^1 \succ_q x^2) \succ_q \dots) \succ_q x^p = \sum_{f \in SH^{\succ}(r_1, \dots, r_p)} q^{s(f)} f \circ (x^1 \times \dots \times x^p),$$

(3) $\gamma^{\prec q}(x^1, \dots, x^p)$ the element:

$$x^1 \prec_q (x^2 \prec_q (\dots (x^{p-1} \prec_q x^p) \dots)) = \sum_{f \in SH^{\prec}(r_1, \dots, r_p)} q^{s(f)} f \circ (x^1 \times \dots \times x^p).$$

Applying the formula of $M_{1n}^{ST(q)}$ to an element $x \setminus y \in \mathbf{Br}$, for $x \in \mathcal{B}_n^r$ and $y = y^1 \times \dots \times y^p$, with $y^j \in \mathbf{Irr}$, we get that:

$$(8.1) \quad \psi^q(x \setminus y) = \sum_{j=0}^p \left(\sum c_{i, i_1, \dots, i_p} \gamma^{\prec q}(y_{i_1}^1, \dots, y_{i_j}^j) \succ_q x^{\perp} \prec_q \gamma^{\succ q}(y_{i_{j+1}}^{j+1}, \dots, y_{i_p}^p) \right),$$

where $\psi^q(x) = \sum_{\perp} a_{\perp} x^{\perp}$, $\psi^q(y^j) = \sum_{b_{i_j} \neq 0} b_{i_j} y_{i_j}^j$, and $c_{i, i_1, \dots, i_p} \in \mathbb{Z}$.

Using Notation 8.3, we rephrase formula (8.1) as follows:

$$(8.2) \quad \psi^q(x \setminus y) = \sum_{j=0}^p \left(\sum c_{i, i_1, \dots, i_p} h \circ (f \circ (y_{i_1}^1 \times \dots \times y_{i_j}^j) \times x^{\perp} \times g \circ (y_{i_{j+1}}^{j+1} \times \dots \times y_{i_p}^p)) \right),$$

where the sum is taken over all $f \in SH^{\prec}(r_{i_1}, \dots, r_{i_j})^q$, $g \in SH^{\succ}(r_{i_{j+1}}, \dots, r_{i_p})^s$ and $h \in SH(u, r, s)$ such that $h(u) \leq h(u+r)$ and $h(u+r) > h(u+r+s)$, for $y_{i_k}^k \in \mathbf{ST}_{n_{i_k}}^{r_{i_k}}$.

In a similar way, if x_1, \dots, x_p is a family of elements in \mathbf{Br} , with $p \geq 2$, and $x = x^1 \cdot \dots \cdot x^p \in \mathbf{Prod}(p)$, then

$$(8.3) \quad \psi^q(x) = \sum \left(\sum_{f \in SH^{\bullet}(r_{i_1}, \dots, r_{i_p})} c_{i_1 \dots i_p} f \circ (x_{i_1}^1 \times \dots \times x_{i_p}^p) \right),$$

where $\psi^q(x^j) = \sum_{a_{i_j} \neq 0} a_{i_j} x_{i_j}^j$, with $x_{i_j}^j \in \mathbf{ST}_{n_j}^{r_{i_j}}$, for $1 \leq j \leq p$.

Reduction to the case $q = 0$

We want to see that if ψ^0 is an isomorphism, then ψ^q is an isomorphism too, for all $q \in \mathbb{K}$. This result implies that the freeness of \mathcal{ST}_{qT} is equivalent to the freeness of \mathcal{ST}_{0T} , for any $q \in \mathbb{K}$.

The proof of the following Lemma is immediate.

8.4. Lemma. *Suppose that x^1, \dots, x^p is a family of surjective maps such that $x^i \in \mathbf{ST}_{n_i}^{r_i}$. For $f \in SH(r_1, \dots, r_p)$, we get that the map $f \circ (x^1 \times \dots \times x^p)$ belongs to \mathbf{ST}_n^s , for $s \leq r_1 + \dots + r_p$. Moreover, $s = r_1 + \dots + r_p$ if, and only if, $f \in Sh(r_1, \dots, r_p)$. In particular, if $f \in SH^{\bullet}(r_1, \dots, r_p)$, then $s < r_1 + \dots + r_p$.*

For all $x \in \mathbf{Irr}_n^r$ such that $\psi^q(x) = \sum_{a_i \neq 0} a_i x_i$, the elements $x_i \in \mathbf{ST}_n^{r_i}$ satisfy that $r_i \leq r$. So, we may restrict ourselves to work with $\sum_{\substack{r_i=r \\ a_i \neq 0}} a_i x_i$ instead of $\psi^q(x)$.

Note that for $x \in \mathcal{B}_n^r$, the element $\psi^q(x) = \sum_{\underline{l}} \alpha_{\underline{l}} x^{\underline{l}}$, with $x^{\underline{l}} \in \mathbf{ST}_n^r$, does not depend on q .

Suppose that $x = y \setminus (z^1 \times \dots \times z^p)$, with $y \in \mathcal{B}_m^s$, $z^j \in \mathbf{Irr}_{n_j}^{l_j}$ such that $\psi^q(z^j) = \sum_{b_{i_j} \neq 0} b_{i_j} z_{i_j}^j$, for $1 \leq j \leq p$. By a recursive argument, we may assume that:

$$\psi^0(z_j) = \sum_{\substack{l_{i_j}=l_j \\ b_{i_j} \neq 0}} b_{i_j} z_{i_j}^j,$$

for $1 \leq j \leq p$. So, $\sum_{\substack{r_i=r \\ a_i \neq 0}} a_i x_i$ is taken over all x_i satisfying that

$$x_i = h \circ (f \circ (z_{i_1}^1 \times \dots \times z_{i_k}^k) \times y^{\underline{l}} \times g \circ (z_{i_{k+1}}^{k+1} \times \dots \times z_{i_p}^p)),$$

with $z_{i_j}^j \in \mathbf{ST}_{n_j}^{l_j}$, for $1 \leq j \leq p$, $f \in \text{Sh}^<(l_1, \dots, l_k)$, $g \in \text{Sh}^>(l_{k+1}, \dots, l_p)$, and $h \in \text{Sh}(l_1 + \dots + l_k, s, l_{k+1} + \dots + l_p)$ such that $h(l_1 + \dots + l_k) < h(l_1 + \dots + l_k + s)$ and $h(l_1 + \dots + l_k + s) > h(l_1 + \dots + l_p + s)$. The coefficient of x_i in $\psi^q(x)$ being $\alpha_{\underline{l}} b_{i_1}^1 \dots b_{i_p}^p$. We get that $\sum_{\substack{r_i=r \\ a_i \neq 0}} a_i x_i = \psi^0(x)$.

In a similar way, if $x = x^1 \cdot \dots \cdot x^p$, with $x^j \in \mathbf{Br}_{n_j}^{r_j}$, $1 \leq j \leq p$, then the unique elements x_i which belong to \mathbf{ST}_n^r are of the form

$$x_i = f \circ (x_{j_1}^1 \times \dots \times x_{j_p}^p),$$

with $x_{j_k}^k \in \mathbf{ST}_{n_k}^{r_k}$, for $1 \leq k \leq p$, and $f \in \text{Sh}^\bullet(r_1, \dots, r_p)$.

As $n_j < n$, we assume that $\psi^0(x^j) = \sum_{\substack{r_{i_j}=r_j \\ b_{i_j} \neq 0}} b_{i_j} x_{i_j}^j$, for $1 \leq j \leq p$. So,

$$\sum_{\substack{r_i=r \\ a_i \neq 0}} a_i x_i = \sum b_{i_1} \dots b_{i_p} \left(\sum_f f \circ (x_{j_1}^1 \times \dots \times x_{j_p}^p) \right),$$

where $\psi^0(x^j) = \sum_{b_{i_j} \neq 0} b_{i_j} x_{i_j}^j$ and $f \in \text{Sh}^\bullet(r_1, \dots, r_p)$, which implies that $\sum_{\substack{r_i=r \\ a_i \neq 0}} a_i x_i = \psi^0(x)$.

The following Proposition is an immediate consequence of the above arguments and of Remark 6.2.

8.5. Proposition. *If $\{\psi^0(x) \mid x \in \mathbf{Irr}_n\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$, then $\{\psi^q(x) \mid x \in \mathbf{Irr}_n\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$ too, for all $n \geq 1$ and all $q \in \mathbb{K}$.*

Note that, even if we have not specified it, the products \cdot^{ST} and M_{1n}^{ST} depend on the tridendriform structure \mathcal{ST}_{qT} considered. From now on, as we have proved that it suffices to work with $q = 0$, the operations \cdot^{ST} and M_{1n}^{ST} will be the ones defined in \mathcal{ST}_{0T} .

Reduction \mathbb{M}

In Lemma 4.7 we proved that for any $x \in \mathbf{ST}_n^r$, and any family $\underline{l} = (l_1, \dots, l_p)$, with $0 = l_0 < l_1 < \dots < l_p < r$, we have that $\mathbb{M}(x^{\underline{l}}) \leq \mathbb{M}(x)$ for the lexicographic order.

We want to prove that, for any $x \in \mathbf{Irr}$ such that $\psi^0(x) = \sum_{a_i \neq 0} a_i x_i$, we have that $\mathbb{M}(x_i) \leq \mathbb{M}(x)$ for the lexicographic order.

Clearly, the result holds for $x \in \mathcal{B}$, since $\psi^0(x) = \sum_{\underline{l}} \alpha_{\underline{l}} x^{\underline{l}}$.

The proof of the following Lemma is easily obtained by mimicking the proof of Lemma 5.9, because the arguments we used to prove it still apply when we replace η by ψ^0 .

8.6. Lemma. *Let $x \in \mathbf{Irr}$ such that $\psi^0(x) = \sum_{a_i \neq 0} a_i x_i$. Any element x_i satisfies that $\mathbb{M}(x_i) \leq \mathbb{M}(x)$ for the lexicographic order. Moreover, for $x = y \setminus (z^1 \times \dots \times z^p)$, with $y \in \mathcal{B}_m^s$ and z^1, \dots, z^p irreducibles, if $\mathbb{M}(x_i) = \mathbb{M}(x)$, then*

$$x_i = h \circ (y^{\underline{l}} \times (g \circ (z_{i_1}^1 \times \dots \times z_{i_p}^p))),$$

for some family \underline{l} such that $\mathbb{M}(y^{\underline{l}}) = \mathbb{M}(y)$, $g \in Sh^{\succ}(r_1, \dots, r_p)$ and $h \in Sh^{\leq}(s, r_1 + \dots + r_p)$, where $z^k \in \mathbf{Irr}_{n_k}^{r_k}$ is such that $\psi^0(z^k) = \sum_{b_{i_k} \neq 0} b_{i_k} z_{i_k}^k$, for $1 \leq k \leq p$.

For an element $x \in \mathbf{Prod}$, we have a similar result.

8.7. Lemma. *Let $x = x^1 \cdot \dots \cdot x^p$ such that $x^i \in \mathbf{Br}_{n_i}^{r_i}$, for $1 \leq i \leq p$. If $\psi^0(x^j) = \sum_{a_{i_j} \neq 0} a_{i_j} x_{i_j}^j$ and $f \in Sh^{\bullet}(r_1, \dots, r_p)$, then*

$$\mathbb{M}(f \circ (x_{i_1}^1 \times \dots \times x_{i_p}^p)) \leq \mathbb{M}(x).$$

The equality holds if, and only if $\mathbb{M}(x_{i_j}^j) = \mathbb{M}(x^j)$, for $1 \leq j \leq p$.

Proof. Denote $w = f \circ (x_{i_1}^1 \times \dots \times x_{i_p}^p)$. It is easily seen that

$$\mathbb{M}(w)_j = \begin{cases} \mathbb{M}(x_{i_p}^p), & \text{for } j = \lambda(w), \\ \mathbb{M}(x_{i_k}^k)_l, & \text{for } j = \lambda(x_{i_1}^1) + \dots + \lambda(x_{i_{k-1}}^{k-1}) + l, \\ \mathbb{M}(x_{i_k}^k)_{\lambda(x_{i_k}^k)} + \mathbb{M}(x_{i_{k+1}}^{k+1}), & \text{for } j = \lambda(x_{i_1}^1) + \dots + \lambda(x_{i_k}^k). \end{cases}$$

The result follows, by applying that $\mathbb{M}(x_{i_k}^k) \leq \mathbb{M}(x^k)$, $1 \leq k \leq p$, and using Lemmas 4.7 and 8.6. \square

So, we get a second reduction. Let

$$\psi_{\mathbb{M}}(x) := \sum_{\mathbb{M}(x_i) = \mathbb{M}(x)} a_i x_i,$$

where $\psi^0(x) = \sum_{a_i \neq 0} a_i x_i$.

8.8. Proposition. *If the set $\{\psi_{\mathbb{M}}(x) \mid x \in \mathbf{Irr}\}$ is linearly independent in \mathcal{ST} , then $\{\psi^0(x) \mid x \in \mathbf{Irr}\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$.*

The image of \mathbf{Br} under $\psi_{\mathbb{M}}$

We want to prove that $\psi^0(\mathbf{Br}(l)) \subseteq \mathbb{K}[\mathbf{Red} \cup \bigcup_{i=0}^l \mathbf{Br}(i)]$, where \mathbf{Red} denotes the set of reducible elements. Our results are similar to the ones of section 5, but in the tridendriform case.

8.9. Lemma. *Let $x \in \mathcal{B}_n^r$, $y \in \mathbf{ST}_m^s$ and $f \in \text{Sh}(r, s)$. If $f \neq \epsilon(r, s)$, then*

$$f \circ (x \times y) \in \bigcup_{l=0}^{m-1} \mathbf{Br}(l).$$

Proof. Suppose that $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$ and $f = \prod_r f'$, with $f' \in \text{Sh}(r-1, s)$. We have that $f \circ (x \times y) = \prod_{j_1 < \dots < j_{\lambda(x)}} f' \circ (x' \times y)$.

If $f \circ (x \times y)$ is reducible, then $f \circ (x \times y) = h^1 \times h^2$, with $|h^1| < j_1$. So,

$$x = f \circ (x \times y)|_{\{1, \dots, n\}} = h^1 \times (h^2|_{\{1, \dots, n-|h^1|\}}),$$

which is false since x is irreducible.

If $f \circ (x \times y) \in \mathbf{Prod}$, then $f' \circ (x' \times y) = h^1 \times h^2$, with $|h^1| < j_{\lambda(x)} - \lambda(x)$. So,

$$x = f \circ (x \times y)|_{\{1, \dots, n\}} = \prod_{j_1 < \dots < j_{\lambda(x)}} h^1 \times (h^2|_{\{1, \dots, n-|h^1|\}}),$$

which does not happen because $x \notin \mathbf{Prod}$.

Suppose that $f \circ (x \times y) \in \mathbf{Br}(l)$, that is $f \circ (x \times y) = z \setminus w$, with $z \in \mathcal{B}$ and $w \in \mathbf{ST}$ with $|w| = l$. Again, we have that

$$x = f \circ (x \times y)|_{\{1, \dots, n\}} = (z \setminus w)|_{\{1, \dots, n\}},$$

which implies that $|z| \geq n$, otherwise, $x = z \setminus (w|_{\{1, \dots, n-|z|\}})$ is not in \mathcal{B} .

We want to see that $|z| > n$. If $|z| = n$, then

$$f \circ (x \times y)(i) = \begin{cases} f(x(i)) = z(i) + p, & \text{for } 1 \leq i \leq n \\ f(y(i-n) + r) = w(i-n), & \text{for } n+1 \leq i \leq n+m, \end{cases}$$

where $w \in \mathbf{ST}_l^p$.

But $x(\{1, \dots, n\}) = \{1, \dots, r\}$ and $y(\{1, \dots, m\}) + r = \{r+1, \dots, r+s\}$, which implies that

$$f(\{1, \dots, r\}) = \{p+1, \dots, p+r\} \text{ and } f(\{r+1, \dots, r+s\}) = \{1, \dots, p\}.$$

As the unique $f \in \text{Sh}(r, s)$ satisfying this condition is $f = \epsilon(r, s)$, the proof is over. \square

8.10. Lemma. *Let $x \in \mathbf{Red}_n^r$, $y \in \mathbf{ST}_m^s$ and $f \in \text{Sh}^{\prec}(r, s)$. The surjection $f \circ (x \times y)$ belongs to $\mathbf{Red} \cup \bigcup_{l=0}^{m-1} \mathbf{Br}(l)$.*

Proof. We have to check that $f \circ (x \times y) \notin \mathbf{Prod} \cup \bigcup_{l \geq m} \mathbf{Br}(l)$.

Let $x = x^1 \times x^2$, for $x^1 \in \mathbf{ST}_{n_1}^{r_1}$ and $x^2 \in \mathbf{Irr}_{n_2}^{r_2}$ such that $x^2 = \prod_{j_1 < \dots < j_{\lambda(x^2)}} x^{2'}$. We have that

$$f \circ (x \times y) = \prod_{j_1 < \dots < j_{\lambda(x^2)}} f' \circ (x^1 \times x^{2'} \times y),$$

with $f' \in \text{Sh}(r-1, s)$.

Suppose that $f \circ (x \times y) \in \mathbf{Prod}$. In this case, $f' \circ (x^1 \times x^{2'} \times y) = h^1 \times h^2$, with $|h^1| < j_{\lambda(x^2)} - k$, which implies that $f'(i) < f'(j)$, for $i \leq j_1$ and all $j \geq r$. Using that $f'(1) < \dots < f'(r-1)$, we get that $f \circ (x \times y)$ is reducible. So, $f \circ (x \times y) \notin \mathbf{Prod}$.

If $f \circ (x \times y) = z \setminus w$, with $z \in \mathcal{B}$, then $|z| > n$ because all the elements of the set $\{f(x(1)), \dots, f(x(n_1))\}$ are smaller than all the elements of the set $\{f(x(n_1+1)), \dots, f(x(n))\}$, and therefore $|w| < m$, which ends the proof. \square

Let $x \in \mathcal{B}_n^r$ and $y = y^1 \times \dots \times y^p \in \mathbf{ST}_m^s$, with $y^j \in \mathbf{Irr}_{m_j}^{s_j}$, $1 \leq j \leq p$. Suppose that $\psi^0(y^j) = \sum_{b_k^j \neq 0} b_k^j y_k^j$, with $y_{k_j}^j \in \mathbf{ST}_{m_j}^{s_j}$.

We proved in Lemma 8.6 that, for $\psi^0(x \setminus y) = \sum_{c_i \neq 0} c_i w_i$, the unique elements w_i such that $\mathbb{M}(w_i) = \mathbb{M}(x \setminus y)$ are of the form

$$w_i = f \circ (x^l \times g \circ (y_{k_1}^1 \times \dots \times y_{k_p}^p)),$$

with $\mathbb{M}(x^l) = \mathbb{M}(x)$, for some $f \in \text{Sh}^{\prec}(r, s)$ and $g \in \text{Sh}^{\succ}(s_1, \dots, s_p)$.

As $x \in \mathcal{B}$, either $x^l = x$ or x^l is reducible. In the second case, applying Lemma 8.10, we get that $w_i \in \mathbf{Red} \cup \bigcup_{l=0}^{m-1} \mathbf{Br}(l)$.

On the other hand, if $f \neq \epsilon_{r,s}$, Lemma 8.9 implies that the element w_i belongs to $\mathbf{Red} \cup \bigcup_{l=0}^{m-1} \mathbf{Br}(l)$.

The argument above proves the following result.

8.11. Proposition. *Let x be an element in \mathcal{B}_n and let $y = y^1 \times \dots \times y^p$ be a surjective map in \mathbf{ST}_n , with y^1, \dots, y^p irreducibles. Suppose that $\psi^0(y^j) = \sum_{b_{k_j}^j \neq 0} b_{k_j}^j y_{k_j}^j$.*

If $\psi_{\mathbb{M}}(x \setminus y) = \sum_{c_i \neq 0} c_i w_i$, then an element w_i which is not of the form

$$w_i = x \setminus (g \circ (y_{k_1}^1 \times \dots \times y_{k_p}^p)),$$

for some $g \in Sh^{\succ}(s_1, \dots, s_p)$, satisfies that $w_i \in \mathbf{Red} \cup \bigcup_{l=0}^{m-1} \mathbf{Br}(l)$.

8.12. Definition. Let $x = y \setminus z$, with $y \in \mathcal{B}_n$ and $z = z^1 \times \dots \times z^p \in \mathbf{ST}_m$, such that $z^j \in \mathbf{Irr}_{m_j}$. Define

$$\psi_{Br}(x) := y \setminus \omega^{\succ}(\psi^0(z^1), \dots, \psi^0(z^p)) = \sum_{c_{i_1 \dots i_k}} \left(\sum_{f \in Sh^{\succ}(s_1, \dots, s_j)} y \setminus f \circ (z_{i_1}^1 \times \dots \times z_{i_p}^p) \right),$$

where $\psi^0(z^j) = \sum_{a_{i_j} \neq 0} a_{i_j} z_{i_j}^j$, and $c_{i_1 \dots i_k} = a_{i_1} \dots a_{i_p}$.

Propositions 8.8 and 8.11 imply the following result:

8.13. Corollary. (1) *The image of \mathbf{Br} under $\psi_{\mathbb{M}}$ is a subspace of $\mathbb{K}[\mathbf{Red} \cup \mathbf{Br}]$.*
 (2) *If the set $\{\psi_{Br}(x) \mid x \in \mathbf{Br}\}$ is linearly independent in $\mathbb{K}[\mathbf{Red} \cup \mathbf{Br}]$, then $\{\psi^0(x) \mid x \in \mathbf{Br}\}$ is linearly independent in $\mathbf{Prim}(ST)$.*

The image of Prod under $\psi_{\mathbb{M}}$

Given elements $x^i = \prod_{j_1^i < \dots < j_{\lambda(x^i)}^i} x^{i'}$, for $1 \leq i \leq p$, recall that

$$x^1 \cdot \dots \cdot x^p = \prod_{j_1^1 < \dots < j_{\lambda(x^p)}^p + n_1 + \dots + n_{p-1}} (x^{1'} \times \dots \times x^{p'}).$$

For $p \geq 2$, $\mathbf{Prod}(p)$ is the set of elements $x = x^1 \cdot \dots \cdot x^p \in \mathbf{Irr}$ such that $x^i \in \mathbf{Br}$, $1 \leq i \leq p$.

We want to give a description of the image

$$\psi_{\mathbb{M}}(x^1 \cdot^{ST} \dots \cdot^{ST} x^p) = \psi_{\mathbb{M}}(x^1) \cdot^{ST} \dots \cdot^{ST} \psi_{\mathbb{M}}(x^p),$$

where $x^1 \cdot^{ST} \dots \cdot^{ST} x^p = \sum_f \prod_{j_1^1 < \dots < j_{\lambda(x^p)}^p + n_1 + \dots + n_{p-1}} f \circ (x^{1'} \times \dots \times x^{p'})$, and the sum is taken over all $f \in \text{Sh}(r_1-1, \dots, r_p-1)$.

The following result completes the ones of Lemmas 8.9 and 8.10.

8.14. Lemma. *Let $x \in \mathbf{Indec}$ and $y \in \mathbf{ST}_m^s$ be surjective maps, and let $f \in \text{Sh}^<(r, s)$. The element $f \circ (x \times y)$ is indecomposable.*

Proof. The result has been proved for $x \in \mathbf{Red} \cup \mathcal{B}$. Suppose that $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$ is indecomposable and $x \notin \mathbf{Red} \cup \mathcal{B}$. We have that:

- (1) $1 < j_1$ and $\lambda(x) < n$,
- (2) if $x' = x^1 \times x^2$, then x^1 is irreducible and $|x^1| > j_{\lambda(x)} - \lambda(x)$,
- (3) $f \circ (x \times y) = \prod_{j_1 < \dots < j_{\lambda(x)}} f' \circ (x' \times y)$, with $f' \in \text{Sh}(r-1, s)$.

Note that $f \circ (x \times y) \notin \mathbf{Indec}$ if, and only if, $f' \circ (x' \times y) = z^1 \times z^2$ with $j_1-1 < |z^1| \leq j_{\lambda(x)} - \lambda(x)$.

As $f'(1) < \dots < f'(r-1)$, we get that

$$x' = \text{std}(f' \circ (x' \times y))|_{\{1, \dots, n-\lambda(x)\}} = \text{std}(z^1 \times z^2)|_{\{1, \dots, n-\lambda(x)\}} = z^1 \times \text{std}(z^2|_{\{1, \dots, n+m-\lambda(x)-|z^1|\}}),$$

which implies $|z^1| \geq n + m - \lambda(x) > j_{\lambda(x)} - \lambda(x)$, in contradiction with the fact that $|z^1| \leq j_{\lambda(x)} - \lambda(x)$. So, $f \circ (x \times y)$ is indecomposable. \square

8.15. Proposition. *For any surjective map $x \in \mathbf{Br}$, the element $\psi_{\mathbb{M}}(x)$ belongs to $\mathbb{K}[\mathbf{Indec}]$.*

Proof. Suppose that $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x' \in \mathcal{B}_n^r$, and that $\underline{l} = (l_1, \dots, l_p)$ is a family of positive integers $1 \leq l_1 < \dots < l_p < r$. The element $x^{\underline{l}}$ is decomposable if, and only if, $x|_{\{l_p+1, \dots, r\}}$ is decomposable.

As $x \in \mathbf{Indec}$, we get that $x^{-1}(\{1, \dots, l_p\}) \cap \{j_1 + 1, \dots, n\} \neq \emptyset$, and therefore Lemma 4.7 implies that $\mathbb{M}(x^{\underline{l}}) < \mathbb{M}(x)$.

Let $x = y \setminus z$ be an element in $\mathbf{Br}(l)$, for $l \geq 1$, and suppose that $\psi_{\mathbb{M}}(x) = \sum_{a_i \neq 0} a_i x_i$. For any i , we know that

$$x_i = h \circ (y^{\underline{l}} \times \bar{z}),$$

for some \underline{l} such that $\mathbb{M}(y^{\underline{l}}) = \mathbb{M}(y)$, some element $\bar{z} \in \mathbf{ST}$ and some $h \in \text{Sh}^<(s, r-s)$, where $y \in \mathcal{B}_m^s$.

As $y^{\underline{l}} \in \mathbf{Indec}$, applying Lemma 8.14 we get that x_i is indecomposable. \square

8.16. Remark. Let $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x'$ be an indecomposable surjection, there exist unique elements y, z and w such that:

- (1) $y, w \in \mathbf{ST} \cup \{1_{\mathbb{K}}\}$ and $z \in \mathbf{Irr}$,

- (2) $x = \prod_{j_1 < \dots < j_{\lambda(x)}} (y \times z \times w)$,
 (3) $0 \leq |y| < j_1 - 1$ and $0 \leq |w| < n - j_{\lambda(x)}$.

If x is irreducible, then $y = 1_{\mathbb{K}}$.

8.17. Example. Let $x = (2, 1, 5, 7, 3, 7, 3, 4, 5, 6)$, we have that $y = (2, 1)$, $z = (3, 1, 1, 2, 3)$ and $w = (1)$.

8.18. Remark. Note that if $f = \prod_{r_1 < \dots < r_1 + \dots + r_p} f'$ and $g = \prod_{r_1 < \dots < r_1 + \dots + r_p} g'$ are two elements in $\text{Sh}^\bullet(r_1, \dots, r_p)$, then $f < g$ for the weak Bruhat order in \mathbf{ST} if, and only if, $f' < g'$ for the weak Bruhat order on $S_{r_1 + \dots + r_p - p}$.

For any positive integer λ and any $\overline{m} = (m_1, \dots, m_p)$, $p \geq 1$, we denote by $\mathbf{ST}_{\lambda, \overline{m}}$ the set of $x \in \mathbf{ST}$ such that $\lambda(x) = \lambda$ and $\mathbb{M}(x) = \overline{m}$. Define the order \leq_{wB} on $\mathbf{Prod}(p)_{\lambda, \overline{m}}$ as the transitive relation spanned by:

$$g \circ (x^1 \times \dots \times x^p) \leq_{wB} f \circ (x^1 \times \dots \times x^p),$$

for any pair of permutations f and g in $\text{Sh}^\bullet(r_1, \dots, r_p)$ such that $f \leq g$ for the weak Bruhat order, where $x^j \in \mathbf{Br}_{n_j}^{r_j}$, $1 \leq j \leq p$.

The order \leq_{wB} is well defined on $\mathbf{Prod}(p)_{\lambda, \overline{m}}$ by Proposition 3.14.

8.19. Proposition. Let x^1, \dots, x^p be a collection of maps, with $x^i \in \mathbf{Indec}_{n_i}^{r_i}$ for $1 \leq i \leq p$, and let $f \in \text{Sh}^\bullet(r_1, \dots, r_p)$.

- (1) If there exist surjections y^1, \dots, y^q such that $f \circ (x^1 \times \dots \times x^p) = y^1 \cdot \dots \cdot y^q$, then $q \leq p$.
 (2) If $f \circ (x^1 \times \dots \times x^p) = y^1 \cdot \dots \cdot y^p$, with $y^j \in \mathbf{Irr}$ for $2 \leq j \leq p$, then one of the following conditions is satisfied:
 (a) $f = \prod_{r_1 < \dots < r_1 + \dots + r_p} 1_{r_1 + \dots + r_p - p}$ and $x^1 \cdot \dots \cdot x^p = y^1 \cdot \dots \cdot y^p$,
 (b) $\prod_{r_1 < \dots < r_1 + \dots + r_p} 1_{r_1 + \dots + r_p - p} < f$ for the weak Bruhat order, and therefore $x^1 \cdot \dots \cdot x^p < y^1 \cdot \dots \cdot y^p$ for the order $<_{wB}$,
 (c) $f = \prod_{r_1 < \dots < r_1 + \dots + r_p} 1_{r_1 + \dots + r_p - p}$ and $(|y^p|, \dots, |y^1|) < (|x^p|, \dots, |x^1|)$ for the lexicographic order.

Proof. For $x^i = \prod_{j_1^i < \dots < j_{\lambda(x^i)}^i} x^{i'}$ and $f_{r_1 < \dots < r_1 + \dots + r_p} = \prod f'$, we get that

$$f \circ (x^1 \times \dots \times x^p) = \prod_{j_1^1 < \dots < j_{\lambda(x^1)}^1 < j_1^2 + n_1 < \dots < j_{\lambda(x^p)}^p + n_1 + \dots + n_{p-1}} f' \circ (x^{1'} \times \dots \times x^{p'}).$$

We proceed by induction on p . For $p = 1$, the result is immediate.

Suppose that $p \geq 2$, and that $y^j = \prod_{l_1^j < \dots < l_{\lambda(y^j)}^j} y^{j'} \in \mathbf{ST}_{m_j}^{s_j}$, for $1 \leq j \leq q$.

I) If $m_q \leq n_p$, then $\lambda(y^q) \leq \lambda(x^p)$. We have to consider two cases:

(1) if $\lambda(y^q) < \lambda(x^p)$, there exists $1 < k_0 \leq n - m_q$ such that:

$$x^p = (y^1 \cdot \dots \cdot y^{q-1})|_{\{k_0, \dots, n-m_q\}} \cdot y^q,$$

which is impossible because x^p is indecomposable.

(2) if $\lambda(y^q) = \lambda(x^p)$, then either $x^p = y^q$, or there exists $z \in \mathbf{ST}_{n_p-m_q}$ such that $x^p = z \times y^q$.

Note that if x^p is irreducible, then the unique possibility is $x^p = y^q$, but we only assume that x^p is indecomposable.

If $x^p = y^q$, then $y^q \in \mathbf{ST}_{n_p}^{r_p}$,

$$f_1 \circ (x^1 \times \dots \times x^{p-1}) = y^1 \cdot \dots \cdot y^{q-1},$$

where $f_1 := f|_{\{1, \dots, r_1 + \dots + r_{p-1}\}} \in \mathbf{Sh}^\bullet(r_1, \dots, r_{p-1})$. Applying a recursive argument, we get that:

- (a) $q \leq p$,
- (b) when $q = p$,
 - (i) if $1_{r_1 + \dots + r_{p-1} - p + 1} < f'_1$, then $x^1 \cdot \dots \cdot x^p < y^1 \cdot \dots \cdot y^p$, because $1_{r_1 + \dots + r_p - p} < f'$.
 - (ii) if $f'_1 = 1_{r_1 + \dots + r_{p-1} - p + 1}$, then $(|y^{p-1}|, \dots, |y^1|) < (|x^{p-1}|, \dots, |x^1|)$. So, we get that $f' = 1_{r_1 + \dots + r_p - p}$ and $(|y^p|, \dots, |y^1|) < (|x^p|, \dots, |x^1|)$.

Otherwise, suppose $x^p = z \times y^q$ is reducible, with $|z| > 0$. We get that $f' = f'|_{\{1, \dots, r - s_p - p + 1\}} \times 1_{s_p - 1}$. There exists a unique way to write down:

$$f'|_{\{1, \dots, r - s_p - p + 1\}} = f_2 \circ (1_{r_1 + \dots + r_{p-2} - p + 2} \times f_3),$$

for a pair of surjective maps $f_2 \in \mathbf{Sh}(r_1 - 1, \dots, r_{p-2} - 1, r_{p-1} + r_p - s_q - 1)$ and $f_3 \in \mathbf{Sh}(r_{p-1} - 1, r_p - s_q)$.

Define the element $\tilde{x}^{p-1} := (\prod_{r_{p-1}} f_3) \circ (x_{p-1} \times z)$. We get that:

$$(\prod_{r_1 < \dots < r_1 + \dots + r_{p-1}} f_2) \circ (x^1 \times \dots \times x^{p-2} \times \tilde{x}^{p-1}) = y^1 \cdot \dots \cdot y^{q-1},$$

with $\prod_{r_1 < \dots < r_1 + \dots + r_{p-1}} f_2 \in \mathbf{Sh}^\bullet(r_1, \dots, r_{p-2}, r_{p-1} + r_p - s_p)$, and $\prod_{r_{p-1}} f_3 \in \mathbf{Sh}^\prec(r_{p-1}, r_p - s_p)$.

From Lemma 8.14, we get that $\tilde{x}^{p-1} \in \mathbf{Indec}$. A recursive argument states that $q \leq p$.

If $q = p$, as $|y^p| < |x^p|$, we get that $(|y^p|, \dots, |y^1|) < (|x^p|, \dots, |x^1|)$.

II) Suppose that $n_p < m_q$. Let $k_0 \leq p-1$ be the minimal integer such that $n_1 + \dots + n_{k_0} > n - m_q$. We have to consider two cases:

(1) when $m_q = n_{k_0} + \dots + n_p$, we get that

$$f|_{\{1, \dots, r_1 + \dots + r_{k_0-1}\}} \circ (x^1 \times \dots \times x^{k_0-1}) = y^1 \cdot \dots \cdot y^{q-1}.$$

Applying a recursive argument, we get that $q-1 \leq k_0-1 < p-1$, so $q < p$ and the result is proved.

(2) when $m_q = l + n_{k_0+1} + \dots + n_p$, for $1 \leq l < n_{k_0}$, as x^{k_0} is indecomposable we get that there $x^{k_0} = \prod_{j_1^{k_0} < \dots < j_{\lambda(x^{k_0})}^{k_0}} x^{k_0} \times z$, for some

$$0 < n_{k_0} + \dots + n_p - m_q = |z| < n_{k_0} - j_{\lambda(x^{k_0})}^{k_0}.$$

Consider the element $\tilde{x}^{k_0} := x^{k_0}|_{\{1, \dots, n_{k_0} + \dots + n_p - m_q\}} \in \mathbf{ST}_{n_{k_0} + \dots + n_p - m_q}^l$, for some $l \geq 1$. We get that

$$f|_{\{1, \dots, r_1 + \dots + r_{k_0-1} + l\}} \circ (x^1 \times \dots \times \tilde{x}^{k_0}) = y^1 \cdot \dots \cdot y^{q-1}.$$

So, $q-1 \leq k_0 \leq -1$ by inductive hypothesis, which implies that if $k_0 < p-1$, then $q < p$ and the proof is over for this case.

To end the proof, assume that $k_0 = p-1$. As y^q is irreducible, we have that:

$$y^{q'} = f'|_{\{r_1 + \dots + r_{p-1} - p - l + 2, \dots, r_1 + \dots + r_{p-p}\}} \circ (z \times x^{p'}),$$

where $1_{l+r_p-1} < f'|_{\{r_1 + \dots + r_{p-1} - p - l + 2, \dots, r_1 + \dots + r_{p-p}\}}$.

Moreover, we get that

$$\prod_{r_1 < \dots < r_1 + \dots + r_{p-1}} f'|_{\{1, \dots, r_1 + \dots + r_{p-1} - l - p + 1\}} \circ (x^1 \times \dots \times \tilde{x}^{p-1}) = y^1 \cdot \dots \cdot y^{q-1}.$$

So, by recursive hypothesis, $q \leq p$, and we have that $1_{r_1 + \dots + r_{p-p}} < f'$, which ends the proof. \square

8.20. Definition. For $x \in \mathbf{Prod}(p)$ such that $\psi_{\mathbb{M}}(x) = \sum_{a_i \neq 0} a_i x_i$, define

$$\psi_{\mathbb{M}1}(x) := \sum_{\substack{x_i \in \mathbf{Prod}(p) \\ a_i \neq 0}} a_i x_i,$$

where the sum is taken over all the terms x_i appearing in $\psi_{\mathbb{M}}(x)$ such that $x_i \in \mathbf{Prod}(p)$.

Proposition 8.19 implies that if the set

$$\{\psi_{\mathbb{M}1}(x) \mid x = x^1 \cdot \dots \cdot x^p, x^j \in \mathbf{Br} \text{ for } 1 \leq j \leq p\}$$

is linearly independent, then $\{\psi_{\mathbb{M}}(x) \mid x = x^1 \cdot \dots \cdot x^p, x^j \in \mathbf{Br} \text{ for } 1 \leq j \leq p\}$ is linearly independent, too.

For the next reduction, we need some additional results.

8.21. Lemma. *Let $x = \prod_{j_1 < \dots < j_{\lambda(x)}} x' \in \mathbf{Br}_n^r$ and let $\underline{l} = \{1 \leq l_1 < \dots < l_p < r\}$ be such that $x^{-1}(\{1, \dots, l_p\}) \subseteq \{1, \dots, j_1 - 1\}$. Given a permutation $f \in \text{Sh}^\sim(l_p, r - l_p)$, the element $f \circ (x|_{\{1, \dots, l\}} \times \dots \times x|_{\{l_{p-1}+1, \dots, l_p\}} \times x|_{\{l+1, \dots, r\}})$ is different to x .*

Proof. In order to simplify notation, we assume that $p = 1$, the proof of the general case is identical.

For $1 \leq l < r - 1$, suppose that the set $x^{-1}(\{1, \dots, l\}) = \{k_1 < \dots < k_s\}$, for $k_s < j_1 - 1$.

Let h_1 be the minimal element such that $q_1 = x(h_1) = \max\{x(\{1, \dots, k_s\})\}$. As x is irreducible, we know that h_1 exists and is smaller than k_s , and there exists at least one $h_2 > k_s$ such that $x(h_2) = q_2 \leq q_1$.

Let $f \in \text{Sh}(l, r - l)$.

If $f(l) < q_1$, then the minimal element of $(f \circ (x|_{\{1, \dots, l\}} \times x|_{\{l+1, \dots, r\}}))^{-1}(q_1)$ is $h_1 + m_1$, where m_1 is the cardinal of $\{k_i \mid k_i > h_1\}$.

So, $f \circ (x|_{\{1, \dots, l\}} \times x|_{\{l+1, \dots, r\}}) \neq x$.

For $f(l) \geq q_1$, let m_2 be the number of elements $1 \leq j < h_2$ such that $x(j) > q_2$, and let $m_3 \geq 1$ be the number of elements $1 \leq i \leq s$ such that $f(x(k_i)) \geq q_1$. It is clear that the cardinal of the set

$\{j \mid 1 \leq j < h_2 \text{ such that } f \circ (x|_{\{1, \dots, l\}} \times x|_{\{l+1, \dots, r\}})(j) > f \circ (x|_{\{1, \dots, l\}} \times x|_{\{l+1, \dots, r\}})(h_2)\}$

is greater or equal to $m_2 + m_3$, which implies that $f \circ (x|_{\{1, \dots, l\}} \times x|_{\{l+1, \dots, r\}})$ is different to x . \square

8.22. Theorem. *Let x^1, \dots, x^p be a family of elements such that $x^j \in \mathbf{Br}_{n_j}^{r_j}$, $1 \leq j \leq p$, and $x = x^1 \cdot \dots \cdot x^p \in \mathbf{Prod}(p)_{\lambda, \overline{m}}$.*

If $\psi_{\mathbb{M}1}(x^1 \cdot \dots \cdot x^p) = \sum_{a_u \neq 0} a_u w_u$, then

- (1) *there does not exist an element w_u such that $w_u <_{wB} x^1 \cdot \dots \cdot x^p$,*
- (2) *if $w_u = y^1 \cdot \dots \cdot y^p \in \mathbf{Prod}(p)$ is a minimal element for \leq_{wB} , and $w_u \neq x^1 \cdot \dots \cdot x^p$, then $(|y^p|, \dots, |y^1|) < (|x^p|, \dots, |x^1|)$ for the lexicographic order.*

Proof. (1) For $p = 1$, the result is clear.

Let $\psi_{\mathbb{M}}(x^k) = \sum_{a_{i_k} \neq 0} a_{i_k} x_{i_k}^k$, for $k = 1, \dots, p$. We need to prove that for any collection of elements $x_{i_1}^1, \dots, x_{i_p}^p$ and any $f \in \text{Sh}^\bullet(r_1, \dots, r_p)$, we have that $f \circ (x_{i_1}^1 \times \dots \times x_{i_p}^p) \not<_{wB} x^1 \cdot \dots \cdot x^p$.

The permutation $f' \in \text{Sh}(r_1 - 1, \dots, r_p - 1)$ may be written in a unique way as

$$f' = f_2 \circ (f_1 \times 1_{r_p - 1}) \in \text{Sh}(r_1 - 1, \dots, r_p - 1),$$

where $f_1 \in \text{Sh}(r_1 - 1, \dots, r_{p-1} - 1)$ and $f_2 \in \text{Sh}(r_1 + \dots + r_{p-1} - p + 1, r_p - 1)$.

Applying recursive hypothesis, we know that

$$\prod_{r_1 < \dots < r_1 + \dots + r_{p-1}} f_1 \circ (x_{i_1}^1 \times \dots \times x_{i_{p-1}}^{p-1}) \not\prec_{wB} x^1 \cdot \dots \cdot x^{p-1}.$$

So, it suffices to prove the result assuming that $x_1 \in \mathbf{Prod}(p-1)$ and $x_2 \in \mathbf{Br}$.

Applying Remark 8.16, the element $w_u = f \circ (x_{i_1}^1 \times x_{i_2}^2)$ satisfies that $x_{i_1}^1 = \prod_{j_1^1 < \dots < j_{\lambda(x^1)}^1} (w^1 \times z^1)$ and $x_{i_2}^2 = \prod_{j_1^2 < \dots < j_{\lambda(x^2)}^2} (y^2 \times w^2)$, where $|z^1| < n_1 - j_{\lambda(x^1)}$, $|y^2| < j_1^2 - 1$ and $f \in \mathbf{Sh}^\bullet(r_1, r_2)$.

We want to prove that in any case, whenever $f \circ (x_{i_1}^1 \times x_{i_2}^2) = \bar{x}^1 \cdot \bar{x}^2$, for \bar{x}^1 in $\mathbf{Prod}(p-1)$ and $\bar{x}^2 \in \mathbf{Br}$, we have $g \circ (\bar{x}^1 \cdot \bar{x}^2) \neq x^1 \cdot x^2$.

For elements $f = \prod_{r_1 < r_1 + r_2} f'$, $w^1 \in \mathbf{ST}_{m_1}^{s_1}$, $z^1 \in \mathbf{ST}_{m_1}^{r_1 - s_1 - 1}$ and $y^2 \in \mathbf{ST}_{m_2}^{s_2}$, we have that.

- (1) If $f'(s_1) > f'(r_1 + s_2)$, then $f \circ (x_{i_1}^1 \times x_{i_2}^2)$ belongs to $\mathbf{Prod}(p-1)$.
- (2) If $f'(r_1) < f'(s_1)$ and $f'(r_1 - 1) < f'(r_1 + s_2)$, then:
 - (a) $\bar{x}^1 = \prod_{j_1^1 < \dots < j_{\lambda(x^1)}^1} (f'_1 \circ (w^1 \times z^1 \times y_1^2) \times y_2^2)$, with $y^2 = y_1^2 \times y_2^2$ and $f'_1 \in \mathbf{Sh}(r_1 - 1, l_1)$,
 - (b) $\bar{x}^2 = \prod_{j_1^2 < \dots < j_{\lambda(x^2)}^2} w^2$.
- (3) If $f'(r_1) < f'(s_1)$ and $f'(r_1 + s_2) < f'(r_1 - 1)$, then $f \circ (x_{i_1}^1 \times x_{i_2}^2)$ belongs to $\mathbf{Prod}(p-1)$.
- (4) If $f'(s_1) < f'(r_1)$ and $f'(r_1 + s_2) < f'(r_1 - 1)$, then
 - (a) $\bar{x}^1 = \prod_{j_1^1 < \dots < j_{\lambda(x^1)}^1} (w^1 \times z_1^1)$, with $z^1 = z_1^1 \times z_2^1$,
 - (b) $\bar{x}^2 = \prod_{j_1^2 < \dots < j_{\lambda(x^2)}^2} f'_2 \circ (z_2^1 \times y^2 \times w^2)$, with $f'_2 \in \mathbf{Sh}(l_2, r_2 - 1)$.
- (5) If $f'(r_1 - 1) < f'(r_1 + s_1)$, then
 - (a) $\bar{x}^1 = \prod_{j_1^1 < \dots < j_{\lambda(x^1)}^1} (w^1 \times f'_3 \circ (z^1 \times y^2))$, with $f'_3 \in \mathbf{Sh}(r_1 - s_1 - 1, s_2)$,
 - (b) $\bar{x}^2 = \prod_{j_1^2 < \dots < j_{\lambda(x^2)}^2} w^2$.

The unique cases where $f \circ (x_{i_1}^1 \times x_{i_2}^2)$ is the product \cdot of p elements are (2), (4) and (5).

Suppose now that we have $\bar{x}^k = \prod_{j_1^k < \dots < j_{\lambda(x^k)}^k} \bar{x}^{k'} \in \mathbf{ST}_{q_k}^{v_k}$, for $k = 1, 2$, and $g \in \mathbf{Sh}^\bullet(v_1, v_2)$.

If $|\bar{x}^1| < n_1$, then \bar{x}^2 must be reducible in order to get $g \circ (\bar{x}^1 \times \bar{x}^2) = x^1 \cdot x^2$ but \bar{x}^2 is irreducible, so in the case (4) there is no solution.

In case (2), we have that if $g \circ (\bar{x}^1 \times \bar{x}^2) = x^1 \cdot x^2$, then $n_2 \leq n_2 - m_2 + |y_2^2|$, but $|y_2^2| < m_2$, and there does not exist a solution.

In (5), we need that $f'_3 \circ (z^1 \times y^2) = t^1 \times t^2$, with $|t^2| = |y^2| = m_2$. But if f'_3 is not the identity, then $f'_3 \circ (z^1 \times y^2)$ cannot be decomposed as $t^1 \times t^2$, with $|t^2| = m_2$.

If f'_3 is the identity, then we need that $x_2 = g_1 \circ (y_2 \times \bar{x}_2)$.

As $x^2 = y \backslash w$ is in \mathbf{Br} , there exists $\underline{l} = (1 \leq l_1 < \dots < l_p)$ such that $y^2 = y|^{\{1, \dots, l_1\}} \times \dots \times y|^{\{l_{p-1}, \dots, l_p\}}$. But, from Lemma 8.21, we get that there does not exist g_1 such that $x^2 = g_1 \circ (y^2 \times \bar{x}^2)$, which ends the proof of point (1).

(2) Applying the same argument that in point (1), it suffices to prove the assertion for $x^1 \in \mathbf{Prod}(p-1)$ and $x^2 \in \mathbf{Br}$.

Suppose that x^1 and x^2 , are such that $\psi_{\mathbb{M}}(x^1) = \sum_{a_i \neq 0} a_i x_i^1$ and $\psi_{\mathbb{M}}(x^2) = \sum_{b_j \neq 0} b_j x_j^2$.

We have that $w_u = f \circ (x_l^1 \times x_k^2) = w_1 \cdot w_2$, for some pair l, k and $f \in \text{Sh}^\bullet(r_1, r_2)$.

Let

$$(1) \ x_l^1 = \prod_{j_1 < \dots < j_{\lambda(x_1)}} (x'_l \times z_1), \text{ for } 0 < |z_1| < n_1 - j_{\lambda(x_1)},$$

$$(2) \ x_k^2 = \prod_{h_1 < \dots < h_{\lambda(x_2)}} (y_2 \times x'_k), \text{ for } 0 \leq |y_2| < h_1 - 1.$$

As $|x_2^k| = |x_2|$, if $|x_2^k| \leq |w_2|$, then

$$w_2 = f|_{\{k+1, \dots, r_1+r_2-2\}} \circ (z_{12} \times x_k^{2'}),$$

where $z_1 = z_{11} \times z_{12}$, $k+1 \leq r-1$ and $f' \in \text{Sh}(r_1-k-1, r_2-1)$.

As w_2 is irreducible, we get that $f' \neq 1_{r_1+r_2-k-2}$ is not the identity, and therefore $x_l^1 \cdot x_k^2 <_{wB} w_1 \cdot w_2$. So, w_u is not minimal. \square

Applying Theorem 8.22, let

$$\psi_{Prod}(x_1 \cdot \dots \cdot x_p) := \sum_{\substack{w_u \text{ minimal for } \leq_{wB} \\ a_u \neq 0}} a_u w_u,$$

where $\psi_{\mathbb{M}}(x_1 \cdot \dots \cdot x_p) = \sum_{a_u \neq 0} a_u w_u$. We have that:

- (1) $\psi_{Prod}(x_1 \cdot \dots \cdot x_p) = x_1 \cdot \dots \cdot x_p + \dots$,
- (2) if $w_u = y_1 \cdot \dots \cdot y_p \in \text{Prod}(p)$ is minimal and different from $x_1 \cdot \dots \cdot x_p$, then $(|y_p|, \dots, |y_1|) < (|x_p|, \dots, |x_1|)$ for the lexicographic order,
- (3) if the set $\{\psi_{Prod}(x) \mid x \in \mathbf{Prod}\}$ is linearly independent, then the $\{\psi_{\mathbb{M}}(x) \mid x \in \mathbf{Prod}\}$ is linearly independent, too.

Finally, applying Proposition 8.19 to $x = x^1 \cdot \dots \cdot x^p \in \mathbf{Prod}(p)$, we get that:

$$\psi_{Prod}(x) = \sum_{a_i \neq 0} a_i w_i,$$

with $w_i = y_i^1 \cdot \dots \cdot y_i^p \in \mathbf{Prod}(p)$ such that $(|y_i^p|, \dots, |y_i^1|) < (|x^p|, \dots, |x^1|)$ for the lexicographic order. So, if the set $\{x^1 \cdot \dots \cdot x^p \mid x^1, \dots, x^p \in \mathbf{Br}\}$ is linearly independent for any integer $p \geq 2$, we get that

$$\{\psi^0(x^1 \cdot \dots \cdot x^p) \mid x^1, \dots, x^p \in \mathbf{Br}, p \geq 2\}$$

is linearly independent, too.

Proof of Theorem 8.1

In the previous section, we have shown that we may restrict ourselves to prove that $\psi^0 : \mathbb{K}[\mathbf{Irr}] \rightarrow \text{Prim}(\mathcal{ST}_{qT})$ is an isomorphism.

We have that \mathbf{Irr} is the disjoint union of \mathbf{Br} and \mathbf{Prod} . Moreover, we have that $E(\mathbf{Irr}) = \text{Prim}(\mathcal{ST})$ by Remark 4.5. Let us describe the framework of our proof.

As $E(x) - x$ belongs to $\mathbb{K}[\mathbf{Red}]$, for all $x \in \mathcal{B}$, it is immediate that $\{\psi_{\mathbb{M}}(x) \mid x \in \mathcal{B}\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$.

Corollary 8.13 shows that $\psi_{\mathbb{M}}(\mathbf{Br}) \subseteq \mathbb{K}[\mathbf{Br}]$. We shall prove first that $\{\psi_{Br}(x) \mid x \in \mathbf{Br}\}$ spans $\mathbb{K}[\mathbf{Br}]$, which implies that both sets $\{\psi_{\mathbb{M}}(x) \mid x \in \mathbf{Br}\}$ and $\{\psi^0(x) \mid x \in \mathbf{Br}\}$ are linearly independent in $\mathbb{K}[\mathbf{Br}]$.

Using the result above, it is immediate to see that the set

$$\psi_{\mathbb{M}}(\mathbf{Br})_n^\bullet := \{\psi_{\mathbb{M}}(x_1) \cdots \psi_{\mathbb{M}}(x_p) \mid x_j \in \mathbf{Br}_{n_j}, 1 \leq p \leq n \text{ and } \sum_{j=1}^p n_j = n\},$$

is linearly independent. To end the proof, we show that the spaces spanned by the sets $\psi_{\mathbb{M}}(\mathbf{Br})_n^\bullet$ and $\psi_{\mathbb{M}}(\mathbf{Irr}_n)$ are equal, for all $n \geq 1$. We proceed by induction on n .

For $n = 1$ and $n = 2$, the result is evident.

For $n \geq 3$, from Corollary 8.13, we know that if $\{\psi_{Br}(x) \mid x \in \mathbf{Br}_m\}$ and $\{\psi^0(x) \mid x \in \mathbf{Prod}_m\}$ are linearly independent for all $m < n$, hence the set $\{\psi^0(x) \mid x \in \mathbf{Irr}_m\}$ is linearly independent in $\mathbb{K}[\mathbf{ST}_m]$, for all $m < n$.

So, the image of the subspace $\bigoplus_{m=1}^{n-1} \mathbb{K}[\mathbf{Irr}_m]$ under ψ^0 spans the subspace $\bigoplus_{m=0}^{n-1} \text{Prim}(\mathcal{ST})_m$.

8.23. Step Br. We proceed as in Section 5.

By induction, we assume that $\{\psi^0(x) \mid x \in \mathbf{Irr}_m, \text{ for } m < n\}$ spans $\text{Prim}(\mathcal{ST})_{<n} := \bigoplus_{j=1}^{n-1} \text{Prim}(\mathcal{ST})_j$.

Applying Proposition 2.8 we get that

$$\bigoplus_{j=1}^{n-1} \mathbb{K}[\mathbf{ST}_j] \subseteq \bigoplus_{j=1}^{n-1} \omega^{\succ}(\text{Prim}(\mathcal{ST})_{<n}^{\otimes j}).$$

So, the set $\bigcup_{x \in \mathcal{B}_{\leq n}} \{x \setminus (\omega^{\succ}(\text{Prim}(\mathcal{ST})_{<n}^{\otimes j}))\}$ spans $\bigcup_{j=0}^n \mathbb{K}[\mathbf{Br}_j]$, which implies that $\{\psi_{Br}(x) \mid x \in \mathbf{Br} \text{ and } |x| \leq n\}$ is linearly independent in $\mathbb{K}[\mathbf{ST}]$.

Therefore, the set $\{\psi^0(x) \mid x \in \mathbf{Br} \text{ and } |x| \leq n\}$ is linearly independent in $\text{Prim}(\mathcal{ST})$.

8.24. Step Prod. We need to prove that for any $p \geq 2$, the set

$$\{x^1 \cdot \dots \cdot x^p \mid x^1, \dots, x^p \in \mathbf{Br}\}$$

is linearly independent, which is obviously true.

Final comment. In order to simplify notation, our definitions of q -dendriform algebra, brace algebra and GV_q algebra were given in the non-graded case. A graded version of these notions is obtained just applying the Koszul sign convention, and our results still hold. In particular, F. Chapoton's operad of K -algebras in [3], which is described by permutohedra, coincides with a differential graded version of 0-tridendriform algebras, such that the degree of the operations \succ and \prec is 1 while the degree of the associative product \cdot is 0. As 0-tridendriform is non symmetric, the unique relation which is modified by Koszul's sign in Definition 2.1 is:

$$x \cdot (y \succ z) = (-1)^{|y|}(x \prec y) \succ z.$$

In this case, the coboundary map of the permutohedra is described as the unique differential map ∂ of degree -1 such that:

- (1) $\partial(x \succ y) = \partial(x) \succ y + (-1)^{|x|}x \succ \partial(y) - (-1)^{|x|+1}x \cdot y,$
- (2) $\partial(x \cdot y) = \partial(x) \cdot y + (-1)^{|x|+1}x \cdot \partial(y),$
- (3) $\partial(x \prec y) = \partial(x) \prec y + (-1)^{|x|}x \prec \partial(y) - (-1)^{|x|}x \cdot y,$

for homogeneous elements $x, y, z \in \mathcal{ST}$.

In this case, our result implies that the GV_0 algebra structure on $\text{Prim}(\mathcal{ST})$ equipped with the coboundary map of the permutohedra, is a free cacti algebra (see for instance [5]) on the base \mathcal{B} .

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